directed angles are regarded as equivalent if they differ only by multiples of a straight angle.

The *addition* of directed angles is defined by the following laws, seen to be consistent with the definition. $\measuredangle l_1, l_2 + \measuredangle l_2, l_3 = \measuredangle l_1, l_3$; $\measuredangle l_1, l_2 + \measuredangle l_3, l_4 = \measuredangle l_1, l_5$, where l_5 is a line so located that $\measuredangle l_2, l_5 = \measuredangle l_3, l_4$.

From these definitions we have the following relations as bases of operations with directed angles.

THEOREM I. $\measuredangle l, l' + \measuredangle l', l = 180°$.

THEOREM II. *If l_1 is parallel to l_1', and l_2 to l_2', then* $\measuredangle l_1, l_2 = \measuredangle l_1', l_2'$. Again, *if l_1 is* perpendicular *to l_1', and l_2 to l_2', then* $\measuredangle l_1, l_2 = \measuredangle l_1', l_2'$.

THEOREM III. *For any four lines* $\measuredangle l_1, l_2 + \measuredangle l_3, l_4 = \measuredangle l_1, l_4 + \measuredangle l_3, l_2$. For, $\measuredangle l_1, l_2 = \measuredangle l_1, l_4 + \measuredangle l_4, l_2$, and $\measuredangle l_3, l_4 = \measuredangle l_3, l_2 + \measuredangle l_2, l_4$.

THEOREM IV. *A necessary and sufficient condition that three points A, B, C lie on a line is that for any other point D we have*

$$\measuredangle ABD = \measuredangle CBD.$$

For, if AB and CB are equally inclined to BD, they coincide, and conversely.

THEOREM V. *The necessary and sufficient condition that four points A, B, C, D lie on a circle is that* $\measuredangle ABD = \measuredangle ACD$.

For this equation means that (*a*) if B and C are on the same side of AD, then $\angle ABD$ and $\angle ACD$ are equal; and (*b*) if B and C are on opposite sides of AD, then $\angle ABD$ is equal to the supplement of $\angle ACD$. Hence the present theorem follows from the theorem quoted in the first paragraph above.

It would be hard to overestimate the importance of this last theorem. Let us illustrate by proving Simson's theorem, using the same notation as previously, but any figure which may be drawn.

Proof. Since PXB, PZB are right angles, P, B, X, Z lie on a circle (in what order we do not know nor care).

Hence, $\measuredangle PXZ = \measuredangle PBZ$.

Similarly, P, X, Y, C are concyclic, and $\measuredangle PXY = \measuredangle PCY$. But $\measuredangle PBX$ is identically the same as $\measuredangle PBA$, and $\measuredangle PCY$ the same as $\measuredangle PCA$. Hence $\measuredangle PXZ = \measuredangle PBA$, and $\measuredangle PZY = \measuredangle PCA$. But since A, B, C are concyclic, $\measuredangle PBA = \measuredangle PCA$, and $\measuredangle PXZ = \measuredangle PXY$, which, by theorem IV, shows that X, Y, Z are collinear.

It is obvious that this is an entirely general proof. It is a little more verbose than need be, in order to bring out the method clearly. We now apply similar methods to a few more well-known theorems.

THEOREM. *If a point is marked on each side of a triangle (or its extension), and the circles drawn, each of which passes through a vertex of the triangle and the points marked on the adjacent sides, these circles pass through a point, and the lines from this point to the marked points make equal angles with the sides.*

Let the triangle be $A_1A_2A_3$ (Fig. 3), let P_1, P_2, P_3 be marked on A_2A_3, A_3A_1, A_1A_2 respectively. Let circles $A_1P_2P_3$, $A_2P_3P_1$ be drawn, and meet at P.

Then

$$\measuredangle PP_2, PP_3 = \measuredangle P_2A_1P_3 = \measuredangle A_3A_1A_2,$$

$$\measuredangle PP_3, PP_1 = \measuredangle P_3A_2P_1 = \measuredangle A_1A_2A_3.$$

Adding,

$$\measuredangle PP_2, PP_1 = \measuredangle A_3A_1, A_1A_2 + \measuredangle A_1A_2, A_2A_3$$
$$= \measuredangle A_3A_1, A_2A_3 = \measuredangle A_1A_3A_2 = \measuredangle P_2A_3P_1.$$

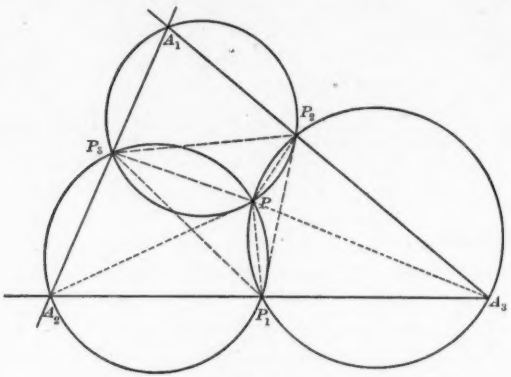


FIG. 3.

Whence, by theorem V, P, P_1, P_2, A_3 are concyclic, and the theorem is proved. Incidentally we see that $\measuredangle PP_1, A_2A_3 = \measuredangle PP_2, A_3A_1 = \measuredangle PP_3, A_1A_2$.

THEOREM. *In the same figure,* $\measuredangle A_2PA_3 = \measuredangle P_2P_1P_3 + \measuredangle A_2A_1A_3$.

The proof consists of splitting $\measuredangle A_2PA_3$ into two parts,

$$\measuredangle A_2PA_3 = \measuredangle A_2PP_1 + \measuredangle P_1PA_3.$$

Now

$$\measuredangle A_2PP_1 = \measuredangle A_2P_3P_1 = \measuredangle A_2A_1P_1 + \measuredangle A_1P_1P_3,$$

and

$$\measuredangle P_1PA_3 = \measuredangle P_1P_2A_3 = \measuredangle P_2P_1A_1 + \measuredangle P_1A_1A_3,$$

and we get the desired result by adding. Of course there are similar expressions for $\measuredangle A_3PA_1$ and $\measuredangle A_1PA_2$. To see the difficulties encountered by attacking this figure without due care, the reader should note McClelland, pages 40–41.

The above is called the theorem of Miquel.[1] The point is called the Miquel point for the set of points P_1, P_2, P_3.

COROLLARIES. (1) If P is a fixed point, it is the Miquel point of infinitely many triangles inscribed in $A_1A_2A_3$. These triangles are all directly similar, with P as center of similitude.

(2) If P lies on the circumcircle of $A_1A_2A_3$, then P_1, P_2, P_3 are collinear, and conversely (Simson's theorem).

[1] Cf. J. L. Coolidge, *Geometry of the Circle*, 1916, p. 85.

For $\measuredangle P_2P_1P_3 = 0$ if and only if $\measuredangle A_2PA_3 = \measuredangle A_2A_1A_3$.

(3) Among the triangles having a given point P for Miquel point is the pedal triangle of P, *i. e.*, the triangle whose vertices are the feet of the perpendiculars from P to the sides of $A_1A_2A_3$. The angles of the pedal triangle of any point are therefore given by the formulas $\measuredangle P_3P_1P_2 = \measuredangle A_2A_1A_3 + \measuredangle A_3PA_2$, etc.

(4) Conversely, if three circles are concurrent at a point, it is possible in an infinite number of ways to draw a triangle having one vertex on each circle and one side passing through each of the intersections of the circles two by two. All such triangles are similar.

Other corollaries suggest themselves readily.

We close with another fundamental theorem, of much less importance, and an application of it to a rather difficult theorem of Steiner.

THEOREM VI. *If O is the center of the circle through A, B, C, then*

$$\measuredangle OAB = \measuredangle ACB + 90°.$$

For, let AO meet the circle again at D. By the rule for adding angles,

$$\measuredangle OAB = \measuredangle ADB + \measuredangle DBA = \measuredangle ACB + 90°.$$

Now corollary 2 above may be re-stated in the following familiar form:

THEOREM. *The circumcircles of the four triangles of a complete quadrilateral meet in a point.*

For if $P_1P_2P_3$ is a transversal of triangle $A_1A_2A_3$, we have seen that the four circles $A_1A_2A_3$, $A_1P_2P_3$, $A_2P_3P_1$, $A_3P_1P_2$ are concurrent. And obviously any complete quadrilateral may be regarded as a triangle and a transversal.

THEOREM (Steiner). *The centers of the four circumcircles lie on a circle which also passes through this point.*

Let P be the intersection of the four circles named above, and let their centers, in the order named, be O, C_1, C_2, C_3. Then $C_1O \perp A_1P$, $C_3O \perp A_3P$, and hence

$$\measuredangle C_1OC_3 = \measuredangle A_1PA_3 = \measuredangle A_1A_2A_3.$$

Similarly for C_2, and we see that the four centers are concyclic. To show that this circle passes through P is not so simple. The triangles C_1PC_3 and $AC_1P_2C_3$ lie symmetrically with regard to C_1C_3; hence

$$\measuredangle C_1PC_3 = \measuredangle C_3P_2C_1 = \measuredangle C_3P_2P_1 + \measuredangle P_3P_2C_1. \quad \text{(addition formula)}$$

But

$$\measuredangle C_3P_2P_1 = \measuredangle P_2A_3P_1 + 90°$$

and

$$\measuredangle P_3P_2C_1 = \measuredangle PA_1P_2 + 90°. \quad \text{(theorem VI)}$$

Hence

$$\measuredangle C_1PC_3 = \measuredangle P_2A_3P_1 + \measuredangle PA_1P_2 = \measuredangle P_3A_1, A_3P_1 = \measuredangle A_1A_2A_3 = \measuredangle C_1OC_3$$

and the proof is completed.

MATHEMATICS IN THE NEW INTERNATIONAL ENCYCLOPEDIA.[1]

By G. A. MILLER, University of Illinois.

The *New International Encyclopedia* has won an eminent position among the American works of reference and is found in a large number of public and private libraries. A second edition has recently appeared in twenty-four large volumes, and the eminence of the men on its editorial staff tends to inspire confidence with respect to accuracy and wise selection of material. The present note does not aim to shake this confidence as regards the useful articles relating to mathematics. On the contrary, it aims to increase the interest in these articles by directing attention to a few desirable corrections and especially to the need of a critical study of some of the statements before accepting them as final authority.

The critical mathematical reader can easily convince himself that certain corrections would greatly improve this standard work. By consulting the article under the word *equation* he will find near the end thereof several surprising statements to the effect that the expressions

$$\sqrt{p^2 - 4q}, \qquad \sqrt{g^2 + 4h^3}$$

are called the discriminants of the two equations

$$x^2 + px + q = 0, \qquad x^3 + 3hx + g = 0$$

respectively. While it is well known that the expression "discriminant of an equation in one unknown" has not always been defined in the same manner by recent noteworthy authors, yet all of these authors seem to agree on the point that such a discriminant is a rational function of the coefficients.[2] The fact that the radical signs noted above are found also in volumes dated 1904 and 1912 respectively of the encyclopedia under consideration makes it more difficult to attribute their appearance in the latest edition to an oversight.

With this clear instance where corrections are needed fresh in mind one may be inclined to approach the rest of this article on the equation in a critical spirit and to observe a considerable number of statements which one would like to change. In fact, the sentence beginning in the eleventh line of this article seems to exhibit too narrow a spirit for a big work. It reads as follows: "The expression $2 + 5 = 7$ expresses an equality, but it is not an equation as the word is technically used in mathematics." If one turns to such a standard work as E. Borel's

[1] This note is a part of a paper read before the Iowa Association of Mathematics Teachers, November 3, 1916.

[2] In the *New Standard Dictionary*, 1915, the following two definitions of the term discriminant are given. "The integral function of the coefficients of an algebraic equation that becomes zero only when the equation has equal roots. The discriminant is equal to the continued product of the squares of all the differences of the roots." Even by means of the quadratic equation $ax^2 + bx + c = 0$ it can easily be verified that these two definitions of the term discriminant are contradictory whenever $a \neq 1$. Two definitions which appear in Webster's *New International Dictionary*, 1916, under the word "discriminant" are contradictory for the same reason.

Die Elemente der Mathematik, translated by P. Stäckel, 1908, one finds on page 7 the following identity

$$145 = 145$$

as an illustrative example of an equation (Gleichung). Many good writers speak of "identical equations" and of "conditional equations," and it would seem that an article in a standard work of reference should recognize this fact.

One does not need to read many lines in the article under consideration before reaching the statement "and in the theory of equations, so called, they [the coefficients] stand for real quantities." It is true that equations with real coefficients usually receive most attention in our elementary works on the theory of equations, but the developments frequently include explicitly the case when the coefficients are complex. In particular, the fundamental theorem of algebra is stated in Dickson's *Elementary Theory of Equations*, 1914, page 47, in the following form: "Every equation with complex coefficients $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n = 0$ has a complex (real or imaginary) root." Hence it would appear that the article in question could be improved by cancelling the statement quoted near the beginning of the present paragraph.

Without trying to direct attention to all the statements in the article under consideration which the critical reader might be inclined to modify, we shall refer to the following, appearing near the end of the first column: "In case there is not a sufficient number of relations given to enable the roots of an equation to be determined, exactly or approximately, the equation is said to be *indeterminate; e. g.*, in the equation $x + 2y = 10$ any of the following pairs of values satisfies the equation: (0, 5), (1, 4.5), (2, 4), (3, 3.5), $\cdots$, (10, 0), (11, $-$ 0.5), $\cdots$." Since such number pairs are called roots of the indeterminate equation in question[1] it is somewhat difficult to see why an indeterminate equation should be characterized as one in which not a sufficient number of relations is given to enable the roots to be determined.

These quotations from a single article may suffice to direct attention to the fact that the reader of the mathematical articles in the *New International Encyclopedia* cannot afford to accept as final all the statements contained therein. It should, however, not be inferred that this particular article, which is also marred by a number of typographical errors, is a fair sample of the mathematical articles in the work under consideration. In view of the importance of the subject treated in this article it seems desirable to endeavor to aid the young reader by directing attention to its shortcomings, especially since such a reader is usually inclined to exercise too little caution in accepting results found in what are commonly regarded as standard works of reference.

In the interesting article on *geometry*, contained in volume 9 of the encyclopedia under consideration, there appears on page 610, column 2, the following perplexing sentence: "Riemann and Helmholtz formulated assumptions for a geometry in space of n-ply manifoldness and with constant curvature and observed that on the sphere, whose curvature is constant and positive, the sum of the

[1] Cf. *Encyclopédie des Sciences Mathématiques*, tome I, Vol. 2, p. 55.

angles of a triangle is less than a straight angle, this characterizing the space of the geometry of Bolyai and Lobachevsky." One might at first be inclined to regard the word "less" in this quotation as due to an oversight and to replace it by the word "greater." If this were done there would evidently be trouble with the rest of the sentence since in the Lobachevsky geometry the sum of the angles of a triangle is actually less than two right angles.

That the article on *determinants*, contained in volume 6 of the work under review, is apt to give an incorrect impression of the fundamental difference between the concepts implied by the terms matrix and determinant, as used by the most careful modern writers, results directly from the following sentences which appear near the beginning of this article: "The first or square form of notation is called the array notation. If there are more columns than rows, the form is called a *matrix*." Probably the critical reader will also be surprised when he meets the following sentence near the close of the article in question: "The theory as a whole has been most systematically treated by F. Brioschi (1824–97), well known as the editor of the *Annali di Matematica*, whose masterly treatise on determinants is a standard (French and German translations, 1856)." F. Brioschi's little treatise was the first book on determinants if we except the monograph by W. Spottiswoode which appeared a little earlier, and it seems very strange that the theory as a whole should be said to have been most systematically treated in the former of these two works when one bears in mind the more modern and more comprehensive treatises by E. Pascal, G. Kowalewski, and others.

The critical mathematical considerations which have been suggested thus far do not relate to what is commonly understood by mathematical history. In view of the emphasis on historical questions in many of the articles in the encyclopedia under review it may be of interest to suggest also a few considerations relating to modifications along this line. To begin with an unusually strong case we may refer to the fact that in the article on *complex number* it is stated that "the first appearance of the imaginary is found in the *Stereometria* of Heron of Alexandria (third century B. C.)." If we turn to the name "Hero or Heron of Alexandria" in the same encyclopedia we meet the following sentence: "The most recent investigation by Schmidt leads to the conclusion that he may have lived in the first century A. D., but other writers, who, it must be said, have not considered the question so fully, have usually placed him in the first or even in the second century B. C." These two statements clearly imply different dates to be associated with the work of an important Greek mathematician. It may be noted in this connection that the same too early date (third century B. C.) for Heron of Alexandria is also found in the *New Standard Dictionary* (1915) and in the *Century Dictionary* (1914).

Another historical statement which seems to need modification appears in the article on *algebra*, in volume 1, page 402, of our encyclopedia, and is as follows: "It was only after the opening of the nineteenth century that Abel, by the use of the theory of groups discovered by Galois, gave the first satisfactory

proof of the fact, anticipated by Gauss and announced by Ruffini, that it is impossible to express the solution of a general equation as algebraic functions of the coefficients when the degree exceeds the fourth." As the first publication by Galois on the theory of groups appeared after the death of Abel it is difficult to see how his proof could have been based upon the discoveries of Galois. At any rate the sentence as quoted above would naturally lead the reader to think that Abel based his work upon work done earlier by Galois and hence it is unsatisfactory in its present form.

We shall refer here to only one more sentence which seems to convey an incorrect impression in regard to a historical fact of considerable mathematical interest. This sentence appears in the article on the Italian mathematician *Cardan*, volume 4, page 536, and is as follows: "The publication of the *Ars Magna* stimulated mathematical research and hastened the general solution of the biquadratic equation, of which Cardan himself had solved special cases." If we read this statement in the light of the fact that Ferrari's general solution of the biquadratic equation actually appeared in the *Ars Magna* it may possibly have some meaning, but it is evident that the beginner would be apt to draw entirely incorrect conclusions therefrom.

The few modifications which have been suggested could scarcely be supposed to be of general interest to mathematics teachers if they did not relate to an excellent work of reference which is very extensively used by college and university students. It is evidently highly desirable that such works be as clear and accurate as possible. We can scarcely expect that the publishers will make special efforts to attain these ends unless the public actually demands them. Hence publicity given to shortcomings, especially where such publicity tends to the discovery of many other important improvements, seems desirable. Such publicity may also tend to inspire caution in the use of even the most reliable works of reference, a caution which needs to be cultivated on the part of most young mathematicians.

NEW RULES OF QUADRATURE.

By P. J. DANIELL, Rice Institute.

The rules given in this paper are developed from Euler's summation formula.[1] This formula has been used in the past chiefly as a means of converting a series into an integral. Nevertheless it has several advantages as a source from which to obtain rules of quadrature. Three such rules are stated here, and the author believes that the second and third are new, while even the first has not received the attention which it deserves.

Rule 1.

$$\int_a^b y dx = h[\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \tfrac{1}{2}y_n] + \frac{h^2}{12}[y_0' - y_n'] + R_1.$$

[1] BROMWICH, *Theory of Infinite Series*, Chap. X, p. 238.

Here the interval (a, b) is divided into n equal parts of length h, y_0, y_1, y_2 $\cdots$ are the ordinates at a, $a + h$, $\cdots$ and y_0', y_n' are the derivatives of y at a and b. If R_1 denotes the remainder, then

$$R_1 = -\frac{1}{4!}\int_a^b y^{IV}(x)\varphi_1(x)dx,$$

where $y^{IV}(x)$ denotes the fourth derivative of y with respect to x, and $\varphi_1(x)$ denotes the function

$$\varphi_1(x) = (x - a - rh)^2(a + rh + h - x)^2, \qquad a + rh \leqq x \leqq a + rh + h.$$

To prove this it is only nccessary to integrate R_1 by parts four times.

$$\int_{a+rh}^{a+rh+h} \varphi_1(x)dx = h^5\int_0^1 t^2(1-t)^2dt = \frac{h^5}{30}.$$

Then

$$|R_1| \leqq \frac{h^4}{6!}(b-a)\max|y^{IV}(x)|.$$

Comparison with Simpson's Rule. To find the error in Simpson's rule, let n be even; then by a similar process, taking $n/2$ intervals of length $2h$,

$$\int_a^b ydx = 2h[\tfrac{1}{2}y_0 + y_2 + y_4 + \cdots + \tfrac{1}{2}y_n] + \frac{4h^2}{12}[y_0' - y_n'] + R_2;$$

where

$$R_2 = -\frac{1}{4!}\int_a^b y^{IV}(x)\varphi_2(x)dx,$$

and

$$\varphi_2(x) = (x - a - 2rh)^2(a + 2rh + 2h - x)^2, \qquad a + 2rh \leqq x \leqq a + 2rh + 2h.$$

Eliminating $[y_0' - y_n']$, we obtain Simpson's rule with remainder R', where $R' = \frac{1}{3}(4R_1 - R_2)$, and

$$\frac{1}{3}\int_{a+2rh}^{a+2r+2h}[\varphi_2(x) - 4\varphi_1(x)]dx = \tfrac{2}{3}h^5\int_0^1 [t^2(2-t)^2 - 4t^2(1-t)^2]dt = \tfrac{4}{15}h^5.$$

Then

$$|R'| \leqq 4\frac{h^4}{6!}(b-a)\max|y^{IV}(x)|.$$

Thus the "maximum error," according to rule 1, is only one quarter of that in Simpson's rule.

Closed Curves. For closed curves Simpson's rule is not applicable except by a separate treatment of different portions. But by an application of rule 1 we can obtain an exceedingly simple rule.

The area of a segment between a chord and the curve, considering the chord as a single interval, will be one twelfth of the square of the chord multiplied by

the change in the tangent of the angle between the curve and the chord. If the chord is small compared to the radius of curvature this change will be approximately the angular change in the direction of the curve. If then we have a number of equal successive chords, the total area between them and the curve will be one twelfth of the square of chord times the total angular change in direction of the curve. For a closed curve this will be 2π, hence we obtain the following:

Rule 2. *Step off equal chords along the curve coming back, if possible, to the starting point. Then if O is any point inside the polygon thus formed,* we have

$$\textit{Area} = \tfrac{1}{2}\ \textit{chord} \times \left[\textit{sum of perpendiculars from } O \textit{ on chords} + \frac{\pi}{3} \times \textit{chord}\right].$$

The first part of this expression is the area of the chord polygon while the latter part is

$$\tfrac{1}{12} \times \text{chord}^2 \times 2\pi = \tfrac{1}{2} \times \text{chord} \times \left[\frac{\pi}{3} \times \text{chord}\right].$$

If the chord polygon is not quite closed we can add to the above expression the area of the remaining sector regarded as a triangle. If the closing chord equals a small fraction λ of the equal chords, then λ times the perpendicular from O on the closing chord is to be added to the sum of the perpendiculars in the rule. Otherwise we may step round again, obtaining the value of twice the area and replacing 2π by 4π. Since the rule is only approximate we may replace $\pi/3$ in the bracket by $1\frac{1}{20}$, thus

$$\textit{Area} = \tfrac{1}{2}\ \textit{chord} \times [\textit{sum of perpendiculars from } 0 + 1\tfrac{1}{20} \textit{ of chord}].$$

Rule 3.

$$\int_a^b y\,dx = \frac{h}{15}(7y_0 + 16y_1 + 14y_2 + 16y_3 + \cdots + 16y_{2n-1} + 7y_{2n}) + \frac{h^2}{15}(y_0' - y_{2n}').$$

Here the interval is divided into an even number $2n$ of parts.

$$\int_a^b y\,dx = h(\tfrac{1}{2}y_0 + y_1 + y_2 + \cdots + \tfrac{1}{2}y_{2n}) + \frac{h^2}{12}(y_0' - y_{2n}') - \frac{h^4}{6!}(y_0''' - y_{2n}''') + S_1,$$

where

$$S_1 = -\frac{1}{6!}\int_a^b y^{\text{VI}}(x)\psi_1(x)\,dx, \qquad \psi_1(x) = z^2(z + \tfrac{1}{2}),$$

and

$$z = (a + rh + h - x)(x - a - rh), \qquad a + rh \leqq x \leqq a + rh + h.$$

To prove this integrate S_1 by parts six times.

Taking n intervals of length $2h$ each,

$$\int_a^b y\,dx = 2h(\tfrac{1}{2}y_0 + y_2 + y_4 + \cdots + \tfrac{1}{2}y_{2n}) + \frac{4h^2}{12}(y_0' - y_{2n}') - \frac{16h^4}{6!}(y_0''' - y_{2n}''') + S_2,$$

where

$$S_2 = -\frac{1}{6!}\int_a^b y^{VI}(x)\psi_2(x)dx, \qquad \psi_2(x) = z_2^2(z_2 + \tfrac{1}{2})$$

and

$$z_2 = (a + 2rh + 2h - x)(x - a - 2rh), \qquad a + 2rh \leqq x \leqq a + 2rh + 2h.$$

Eliminating $[y_0''' - y_{2n}''']$ between these two, we obtain rule 3 with remainder $S' = \frac{1}{15}(16S_1 - S_2)$, and

$$\frac{1}{15}\int_{a+2rh}^{a+2rh+2h} [\psi_2(x) - 16\psi_1(x)]dx$$

$$= \frac{2}{15}h^7\int_0^1 [t^2(2-t)^2(\tfrac{1}{2} + 2t - t^2) - 16t^2(1-t)^2(\tfrac{1}{2} + t - t^2)]dt$$

$$= \frac{4}{7 \times 25} \times 2h^7.$$

Hence

$$|S'| \leqq \frac{32}{25}\frac{h^6}{8!}(b-a)\max|y^{VI}(x)|.$$

Examples. (1) Using Rule 2, to find π by means of the regular polygon of 24 sides inscribed in a circle of unit radius, we have

$$\text{chord} = 2\sin 7°.5, \qquad \text{and} \qquad \text{perpendicular} = \cos 7°.5.$$

Hence,

$$\pi = \sin 7°.5\,[24\cos 7°.5 + 21/20 \times 2\sin 7°.5] = 12\sin 15° + 1.05(1 - \cos 15°).$$

Using

$$\sin 15° = \tfrac{1}{4}(\sqrt{6} - \sqrt{2}), \qquad \text{and} \qquad \cos 15° = \tfrac{1}{4}(\sqrt{6} + \sqrt{2})$$

we find $\pi = 3.14162$ instead of 3.14159.

(2) To approximate the value of $\log_e 2$. We have

$$\int_1^2 \frac{dx}{x} = \log_e 2 = .69315.$$

Using two intervals we get from $\int_1^2 \frac{dx}{x}$ by Simpson's rule, .69444, with the error .00129; by Rule 1, .69271, with the error .00044; and by Rule 3, .69306, with the error .00009.

(3) To approximate the value of π from $\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\,dx$. We have

$$\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\,dx = \frac{\pi}{12} + \frac{1}{8}\sqrt{3}.$$

Using two intervals we get the value of π from $\int_0^1 \sqrt{1-x^2}\,dx$ by Simpson's rule, 3.14093, with the error .00066; by Rule 1, 3.14178, with the error .00019; and by Rule 3, 3.14161, with the error .00002.

SOME METRICAL PROPERTIES OF THE PENTAHEDROID IN A SPACE OF FOUR DIMENSIONS.

By M. H. SZNYTER, University of California.

The purpose of this paper is to establish for the pentahedroid in a four-dimensional space, theorems similar to those found in ordinary solid geometry, dealing with the tetrahedron. The terms line, point and plane are used with the same significance as in three-dimensional geometry. By hyperplane we shall mean that three-dimensional element which consists of any four points not points of one plane, all points collinear with any two of them or with any two obtained by this process.[1] We shall first develop the simpler theorems, then we shall consider the pentahedroid with its tangent hyperspheres.

THEOREM 1.[2] *Let $A_1A_2A_3A_4A_5$ be a pentahedroid cut by a hyperplane α in such a way that the edge A_1A_2 lies on one side of α and the face $A_3A_4A_5$ on the other side. Then the following cases will appear.*

1. *If α is parallel to the line A_1A_2 and to the plane $A_3A_4A_5$, the section will be a prism.*
2. *If α is parallel to the line A_1A_2 but not to the plane $A_3A_4A_5$, the section will be a truncated prism.*
3. *If α is parallel to the plane $A_3A_4A_5$ but not to the line A_1A_2, the section will be a frustrum of a pyramid.*
4. *If α is parallel neither to the plane $A_3A_4A_5$ nor to the line A_1A_2, the section will be a truncated pyramid.*

For the hyperplane α cuts the tetrahedrons $A_2A_3A_4A_5$ and $A_1A_3A_4A_5$ in planes which cut a triangle from each $A_3'A_4'A_5'$ and $A_3''A_4''A_5''$. Since α passes between the edges A_1A_2 and the edges A_3A_4, A_4A_5 and A_3A_5, it must cut the other three tetrahedrons in planes which cut quadrilaterals from them. The section cut out from the pentahedroid will be a three-dimensional figure having for its boundaries two triangular faces and three quadrilateral faces. In cases 1 and 2 the lateral edges of the section are parallel and the section is prismatic. In case 1 the triangular faces are parallel while in case 2 they are not. Hence the former gives a prism while the latter gives a truncated prism. In cases 3 and 4, the lateral edges of the section will meet, if produced, at the point P which is the point of intersection of A_1A_2 with α, and the section is pyramidal. When α is parallel to plane $A_3A_4A_5$, the triangles $A_3'A_4'A_5'$ and $A_3''A_4''A_5''$ are parallel;

[1] MANNING: *Geometry of Four Dimensions*, p. 24.
[2] *Ibid.*, p. 228.

the section is thus a frustrum of a pyramid, as in case 3. When α is not parallel to the plane $A_3A_4A_5$, the planes of the triangles $A_3'A_4'A_5'$, $A_3''A_4''A_5''$ meet if produced in the line g, the intersection of α and $A_3A_4A_5$. This section is therefore a truncated pyramid.

THEOREM 2. *The lateral volumes of two similar pentahedroids are to each other as the cubes on their homologous sides.*

THEOREM 3. *Pentahedroids are to each other as the products of their hyperplane bases by their altitudes; hence pentahedroids which have equivalent bases are to each other as their altitudes and pentahedroids which have equivalent altitudes are to each other as their bases.*

THEOREM 4. *Pentahedroids which have a hyperspace angle of one equal to a hyperspace angle of the other are to each other as the products of the edges of the equal hyperspace angles.*

From theorems 3 and 4 we get at once,

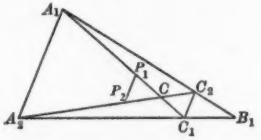
THEOREM 5. *Similar pentahedroids are to each other as the fourth powers of their homologous edges.*

THEOREM 6. *The lines drawn from the five vertices of a pentahedroid to the centers of gravity of the opposite cells cut each other in two segments in the ratio of 4 : 1.*

Consider the pentahedroid $A_1A_2A_3A_4A_5$ with C_k the center of gravity of the cell opposed to the vertex A_k. Since we know that the segments A_1C_1, A_2C_2, $\cdots$ are concurrent at the center of gravity of the pentahedroid we may denote this point by C. The following relations shall hold:

$$A_1C : CC_1 = 4 : 1, \quad A_2C : CC_2 = 4 : 1, \quad \cdots.$$

For upon the segment A_1C_1 consider a point P_1 on the same side of C as A_1 and such that CP_1 equals $\frac{1}{4}CA_1$. Consider four other points P_k similarly situated upon the other four segments A_kC_k.



Then the segments A_1C_2 and A_2C_1 meet at a point B_1 which is the center of gravity of the face $A_3A_4A_5$. For C_2 is the center of gravity of the cell $A_1A_3A_4A_5$ and hence A_1C_2 passes through the center of gravity of the face $A_3A_4A_5$. Likewise C_1 is the center of gravity of the cell $A_2A_3A_4A_5$ and hence A_2C_1 also passes through the center of gravity of the face $A_3A_4A_5$. We now have two triangles $A_1A_2B_1$ and A_1A_2C in the plane determined by A_1B_1 and A_2B_1.

The segment C_1C_2 is parallel to A_1A_2 and is equal to $\frac{1}{4}A_1A_2$, for

$$C_1B_1 = \tfrac{1}{4}A_2B_1 \quad \text{and} \quad C_2B_1 = \tfrac{1}{4}A_1B_1.$$

Similarly the segment P_1P_2 is parallel to A_1A_2 and is equal to $\frac{1}{4}A_1A_2$. Thus

P_1P_2 and C_1C_2 determine a parallelogram, from which relation we obtain at once,

$$CC_1 = P_1C = \tfrac{1}{4}A_1C \quad \text{and} \quad CC_2 = P_2C = \tfrac{1}{4}A_2C$$

or $A_1C : CC_1 = 4 : 1$, etc., which proves our theorem.

PROBLEM 1. *To find the radius of the inscribed and of the circumscribed hyperspheres of any regular pentahedroid in terms of its edge.*

Consider the regular pentahedroid $A_1A_2A_3A_4A_5$ with an edge equal to a certain finite magnitude K. Its ten edges, ten faces, dihedral angles and cells are all equal among each other for we know that the regular pentahedroid is congruent to itself in sixty different ways. It is first necessary to determine the radii of circles inscribed in and circumscribed about the plane faces and also the radii of the spheres inscribed and circumscribed about the five cells of the pentahedroid.

Consider one of the regular faces, $A_1A_2A_3$. By plane geometry the altitude of this triangle is

$$a = \sqrt{K^2 - \tfrac{1}{4}K^2} = \frac{\sqrt{3}}{2}K.$$

Since the three altitudes intersect at the center of gravity of this triangle, the radius of the inscribed circle is one third of the altitude and the radius of the circumscribed circle is two thirds of the altitude. Thus

$$r_i = \frac{1}{3}\frac{\sqrt{3}}{2}K = \frac{\sqrt{3}}{6}K, \quad r_c = \frac{2}{3}\frac{\sqrt{3}}{2}K = \frac{\sqrt{3}}{3}K.$$

Consider one of the regular cells as $A_1A_2A_3A_4$. Its altitude A is measured by the length of one leg of a right-angled triangle whose hypotenuse is the edge K and whose other leg is the radius of the circle circumscribed about the regular face $A_1A_2A_3$. Hence

$$A = \sqrt{K^2 - \frac{3K^2}{9}} = \frac{\sqrt{6}}{3}K.$$

The altitudes of the tetrahedron meet at its center of gravity and hence the radius of the inscribed sphere is one fourth the length of the altitude of the cell and the radius of the circumscribed sphere is three fourths the altitude. Therefore,

$$R_i = \frac{1}{4}\frac{\sqrt{6}K}{3} = \frac{\sqrt{6}}{12}K, \qquad R_c = \frac{3}{4}\frac{\sqrt{6}}{3}K = \frac{\sqrt{6}}{4}K.$$

In the pentahedroid $A_1A_2A_3A_4A_5$, the altitudes are concurrent at the center of gravity of the pentahedroid, therefore the radius of the inscribed hypersphere is equal to one fifth of an altitude and the radius of the circumscribed hypersphere is equal to four fifths of an altitude. Let α represent any altitude of the pentahedroid and ρ_i, ρ_c the radii of the inscribed and circumscribed hyperspheres respectively. As α is measured by the length of one leg of a right-angled

triangle whose hypotenuse is the edge K and whose other leg is the radius R_c of the sphere circumscribed about one of its cells, therefore,

$$\alpha = \sqrt{K^2 - \frac{6}{16}K^2} = \frac{\sqrt{10}}{4}K$$

and

$$\rho_1 = \frac{1}{5}\alpha = \frac{\sqrt{10}}{20}K,$$

radius of the inscribed hypersphere

$$\rho_c = \frac{4}{5}\alpha = \frac{\sqrt{10}}{5}K,$$

radius of the circumscribed hypersphere.

Turning now to the second part of our problem, namely the tangent hyperspheres of the pentahedroid, it is first necessary to establish a system of coördinates for any given point of hyperspace with reference to a fixed pentahedroid. This system may be called pentahedroidal and will be based upon the distances of any point in hyperspace from the five cells of the pentahedroid. It is necessary to find a relation between these distances. They will then be sufficient to determine a single point in hyperspace.

Consider the pentahedroid $A_1A_2A_3A_4A_5$. Let V_k be the volume of the cell opposite the summit A_k and x_k the distance of any point P from the hyperplane V_k. Then the necessary and sufficient condition that any five quantities x_1, x_2, x_3, x_4, x_5 represent the coördinates of the point P with respect to the pentahedroid is that they satisfy the following equation,

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H, \qquad (1)$$

where H is the hypervolume of the pentahedroid. The five hyperplanes of the pentahedroid divide hyperspace into thirty-one regions, as follows:

1. The interior $A_1A_2A_3A_4A_5$ of the pentahedroid.
2. The region α having only the vertex A_1 in common with the pentahedroid and lying on the same side of $A_2A_3A_4A_5$ that A_1 lies. There are five regions of this sort, one opposed to each vertex of the pentahedroid.
3. The region β bounded by three hyperplanes having the edge A_2A_3 in common and two hyperplanes having each only one point in common with the edge A_2A_3. There are ten regions of this sort, one opposite each edge of the pentahedroid.
4. The region γ bounded by two hyperplanes having the face $A_2A_3A_4$ in common and by three hyperplanes having each only a line in common with the face $A_2A_3A_4$. There are ten regions of this sort, one relative to each face.
5. The region δ formed by removing the pentahedroid from the interior of one of its tetrahedroidal angles. There are five such regions formed by taking each of the points A_1, A_2, $\cdots$ in turn.

Then for any point lying in any of the regions so determined, the perpendicular distances from it to the five hyperplanes of the pentahedroid shall satisfy equation (1).

Consider P any point within the pentahedroid and x_1, x_2, x_3, x_4, x_5 the distances from this point to the five cells of the pentahedroid which is now divided into five smaller pentahedroids formed by the five cells taken in turn with the point P as a common vertex. The hypervolume of the original pentahedroid is equal to the sum of the hypervolumes of the five smaller pentahedroids. Since the hypervolume of any pentahedroid is equal to one fourth the product of its hyperplane base by its altitude[1] we have at once,

$$4H = V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5.$$

If the point P is within the region α, the pentahedroid $A_1A_2A_3A_4A_5$ is equal to the pentahedroid $PA_2A_3A_4A_5$ minus the sum of the other four pentahedroids determined by P with the other four cells of the original pentahedroid. The altitudes of the five pentahedroids having P for a vertex are x_1, $-x_2$, $-x_3$, $-x_4$, $-x_5$.[2] Hence

$$4H = V_1x_1 - (-V_2x_2) - (-V_3x_3) - (-V_4x_4) - (-V_5x_5)$$

or

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

When P is within the region β, the five altitudes of the five pentahedroids determined by P are $-x_1, x_2, x_3, -x_4$ and $-x_5$. Thus

$$4H = V_2x_2 + V_3x_3 - (-V_4x_4) - (-V_5x_5) - (-V_1x_1),$$

or

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

For P in the region γ the altitudes of the five pentahedroids determined by it are $-x_1, x_2, x_3, x_4, -x_5$ and for ρ in the region δ the altitudes are $-x_1, x_2, x_3, x_4$, and x_5. The relations between the hypervolumes become

$$4H = V_2x_2 + V_3x_3 + V_4x_4 - (-V_1x_1) - (-V_5x_5) \text{ for } \gamma,$$

$$4H = V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 - (-V_1x_1) \qquad \text{for } \delta,$$

whence

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

From the above consideration it is evident that if a point P lies within any one of the regions determined by the five hyperplanes of a pentahedroid, its distances to the five hyperplanes satisfy a certain relation which is expressed by equation (1). To complete the establishment of the system it is necessary to show that if any five quantities x_1, x_2, x_3, x_4 and x_5 satisfy equation (1) they determine one and only one point in hyperspace.

[1] MANNING: *Geometry of Four Dimensions*, pp. 277–278.

[2] Any distance x_k shall be positive or negative depending on whether it is on the same side or opposite side of the hyperplane V_k that the vertex A_k is.

Consider the hyperplanes P_1, P_2, P_3, P_4 and P_5 at the distances x_1, x_2, x_3, x_4 and x_5 from the hyperplanes V_1, V_2, V_3, V_4 and V_5 and such that P_k is parallel to V_k. Any four hyperplanes will meet at a point in hyperspace.

Suppose P_1, P_2, P_3 and P_4 meet at Q and let g_1, g_2, g_3, g_4 and g_5 be the coördinates of the point φ with respect to the pentahedroid. From (1) we have

$$V_1g_1 + V_2g_2 + V_3g_3 + V_4g_4 + V_5g_5 = 4H.$$

From the construction of the hyperplanes P_1, P_2, P_3, P_4, we have

$$g_1 = x_1, \quad g_2 = x_2, \quad g_3 = x_3, \quad g_4 = x_4.$$

Therefore

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5g_5 = 4H.$$

Also

$$V_1x_1 + V_2x_2 + V_3x_3 + V_4x_4 + V_5x_5 = 4H.$$

Therefore $g_5 = x_5$ and Q is a single point of hyperspace.

THEOREM 7. *There are at most sixteen hyperspheres tangent to the five cells of a pentahedroid.*

1. If there be a hypersphere inscribed within the pentahedroid, the coördinates of the center C, besides being all equal, must satisfy equation (1). If R be the radius of the hypersphere, then

$$x_1 = x_2 = x_3 = x_4 = x_5 = R,$$

and hence

$$RV_1 + RV_2 + RV_3 + RV_4 + RV_5 = 4H,$$

$$R(V_1 + V_2 + V_3 + V_4 + V_5) = 4H,$$

$$R = \frac{4H}{V_1 + V_2 + V_3 + V_4 + V_5}. \tag{2}$$

Since H, V_1, V_2, V_3, V_4, V_5 are all positive, (2) gives a single positive value of R, hence there is a single hypersphere inscribed in the pentahedroid, tangent to its five cells.

2. If there exists a hypersphere tangent to the cells with its center within the region α, the coördinates of the center will satisfy the following,

$$x_1 = -x_2 = -x_3 = -x_4 = -x_5 = R$$

and

$$RV_1 - RV_2 - RV_3 - RV_4 - RV_5 = 4H,$$

whence

$$R = \frac{4H}{V_1 - (V_2 + V_3 + V_4 + V_5)}. \tag{3}$$

As any four cells of the pentahedroid are together greater than the fifth, the right-hand member of (3) is negative, and since a negative value of R is inadmissible, there is no hypersphere tangent to the cells with its center in the region α. Therefore there are no tangent hyperspheres in the other four regions of the type α.

3. If there be tangent hyperspheres in the regions β and γ, the following must hold:

$$-x_1 = x_2 = x_3 = -x_4 = -x_5 = R,$$

$$-x_1 = x_2 = x_3 = \quad x_4 = -x_5 = R,$$

and

$$R = \frac{4H}{V_2 + V_3 - (V_1 + V_4 + V_5)} \text{ in } \beta, \tag{4}$$

$$R = \frac{4H}{V_2 + V_3 + V_4 - (V_1 + V_5)} \text{ in } \gamma. \tag{5}$$

The value of R is now dependent on whether the sum of two cells of a pentahedroid is less than, equal to or greater than the sum of the other three cells. Thus if $V_2 + V_3 - (V_1 + V_4 + V_5)$ is negative in (4) then $V_1 + V_4 + V_5 - (V_2 + V_3)$ in (5) will be positive giving a positive R in γ for a negative value of R in β. Thus the number of tangent hyperspheres in the regions β and γ together will always be ten unless some of these expressions happen to be zero.

4. Finally when the center of the tangent hypersphere lies within the region δ,

$$-x_1 = x_2 = x_3 = x_4 = x_5 = R,$$

$$R = \frac{4H}{V_2 + V_3 + V_4 + V_5 - V_1}. \tag{6}$$

As the sum of any four cells of a pentahedroid is greater than the fifth, (6) gives a single positive value for R and hence there is one hypersphere tangent to the cells with its center in the region δ. There are five such hyperspheres, one in each of the five regions of the type δ.

Thus we have one tangent hypersphere within the pentahedroid, ten within the regions β and γ taken together and five in the regions δ, making the number of tangent hyperspheres equal in general to sixteen.

BOOK REVIEWS.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

Plane and Solid Geometry. By WEBSTER WELLS and WALTER W. HART. D. C. Heath and Company, Boston, 1916. viii + 467 pages.

It is very important that the introduction to a course in plane geometry should be well done. With inexperienced teachers, at least, the introduction to the course will usually be the one that is given in the text. The introductory chapter of a geometry may be expected to make a clear statement of certain definitions and assumptions and to illustrate them; to offer some suggestions that will tend to make the students feel a need for geometry; and to show the need of formal proofs and to give an introduction to them. In the present text

the definitions are well illustrated with simple exercises. Not much is done in the introduction to show the uses of geometry. More exercises in construction would make the introduction more interesting and more effective in fixing new ideas.

The following sentences from the preface suggest the attitude of the authors in the selection and the arrangement of material: "In each Book, the fundamentally important theorems are given first. These theorems present a *safe and sane minimum course.* These are followed in each Book by one or more groups of theorems or applications which are strictly supplementary—material which has either long appeared in geometries in some form or has been introduced in recent years to add to the pupils' interest."

Care has been taken to make the book teachable. Teachers examining this text will be interested in the summaries such as appear on pages 27, 62, and 113; in the references to supplementary exercises in the footnotes; in the treatment of loci and inequalities; and in the suggested proofs which are left for the students to finish. The exercises are numerous and appear to be well graded. There is a table of trigonometric ratios and some problems applying them.

Certain definitions need revision. The definition of a circle given in § 16 is: "A circle is a closed curve all points of which are equidistant from a point within called the center." The circle need not be a plane figure by this definition. The definition of ratio, § 212, reads as follows: "The ratio of two magnitudes of the same kind is the quotient of their numerical measures in terms of a common measure." It seems likely that students will be confused by this definition. The following definition of the limit of a variable is given in § 401: "A limit of a variable is a constant such that the numerical value of the difference between the constant and the variable becomes and remains less than any small positive number." The "small positive number" must of course be pre-assigned.

In three important respects the book has departed little from the traditional. The exercises are mainly formal, with a comparatively small number of applications; not much has been done in applying algebra to the solution of geometric problems; and the notions of motion and symmetry, which are useful and which have become prominent in the teaching of elementary geometry in recent years, have received slight attention.

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Fundamental Conceptions of Modern Mathematics: Variables and Quantities, with a Discussion of the General Conception of the Functional Relation. By ROBERT T. RICHARDSON and EDWARD H. LANDIS. Chicago and London, The Open Court Publishing Co., 1916.

In their preface the authors describe their work as "essentially one of constructive criticism . . . the first attempt made on any extensive scale to examine critically the fundamental conceptions of mathematics as embodied in the current definitions." The present volume, which is to be followed by a second, is concerned with variables and quantities.

No one interested in recent efforts to bring precision and order into the conceptions of mathematics can fail to commend the purpose of the joint authors or to agree with them in their view of the unsatisfactory character of the usual definitions. On the other hand, the book fails, I think, to present any convincing and novel constructive results. The ideas of the authors are not presented either with the clearness or the acumen necessary to make them effective and are not as new as the authors suppose. For example, the main thesis of the authors regarding functional relation, to wit, that it often involves likeness of order of the corresponding quantities of the variables has been expressed with far greater exactness and logical completeness by Russell in his "Principles of Mathematics," Chapter XXXII (On the Correlation of Series); while, on the other hand, the argument (pages 151, 152) by means of which they seek to restrict the notions of variable and function to relations between *ordered* aggregates is not likely to convince "the Peano school" against whom it is directed. Philosophically speaking, a variable denotes *any* term standing in a given relation and a function denotes the relation in question. The definitions of Richardson and Landis simply restrict the broader definitions to more special cases. But the broader definitions should be insisted upon for the sake of the logical unification which they confer upon the various branches of mathematics. And when it comes to the point of giving a constructive definition of variable, the authors resign the effort on the plea that it would require a too lengthy discussion of the philosophical doctrine of the categories (page 154 et seq.).

The consideration of quantity yields an equally disappointing result. Of all the concepts of mathematics requiring clarification, this is the most flagrant; yet, so far as I can see, the present authors only add to the confusion.

More in accord with what the reviewer would regard as sound doctrine is the stand taken against the symbol theory of pure mathematics. For, even before any real objects satisfying the variables in a mathematical expression have been found, the expression is not a mere symbol, but indicates certain operations or relations between *possible* objects. Good remarks are also to be found concerning the distinction between independent and dependent variables (page 175).

Taken as a whole, what the book lacks is exactly what it aims to embody—a philosophical background for the criticism of mathematical ideas. One representative illustration of this is the naïve discussion of similarity and identity on pages 5 and 6. But after all, this is only the first and preliminary volume.

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The Arithmetical Philosophy of Nicomachus of Gerasa. By GEORGE JOHNSON, Ph.D. (University of Pennsylvania dissertation.) New Era Printing Company, Lancaster, Pa. 1916. 49 pages.

This pamphlet presents a partial translation (the complete translation is deposited in the library of the University of Pennsylvania) of the famous *Intro-*

duction to Arithmetic of Nicomachus, the earliest extant separate essay at a systematization of the Greek science of Arithmetikê, treating of numbers as such, as distinguished from Logistikê, which treated of numerical reckoning from a practical standpoint. Although the *Introduction* was no such landmark in the history of mathematical advance as the works of Euclid, or Archimedes, or Diophantus, or of some few other Greeks, it is still an extremely important document in educational mathematics and continued to exercise a great influence in the Latin versions of Apuleius and Boethius.

Dr. Johnson's dissertation is not a pretentious performance, and its aim seems to be description only, for he has not added to the translation, which covers a selection of passages designed to show the general argument of the *Introduction*, any independent treatment of Nicomachus himself, his philosophy, or his number theory. The commentary which he presents is confined to summaries and explanatory matter connecting the translated passages (for the footnote to the title is the only one in the book), and is largely based on Nesselmann and Cantor. Recognizing the generally descriptive nature of the commentary, the reader will still find statements that need modification. It does not give a true idea of the situation to say that the *Theologumena Arithmetica*, which as it stands is a patchwork of passages from several hands, is "probably the work of Nicomachus" (p. 1), although it certainly is largely Nicomachean; nor does it mean much to say that Nicomachus, long before algebra was developed, was "among the first to attempt a systematic treatment of arithmetic distinct from algebra" (p. 2). The statement that the number contrasted by Nicomachus with his "scientific" number is the "ideal number" of Plato (p. 6) surely calls for proof, for the divine numbers which are discussed in the *Theologumena Arithmetica* ought to be brought into consideration, as well as the Platonic. Some of the formulas, also, might be criticized. $(2n - 1)/n$, for example, hardly comprehends all possible superpartials (defined in the treatise as the relation between two numbers "whenever a number contains in itself the whole compared and in addition more of its parts than one," I. 20. 1), and it is a bad slip to give $a : \sqrt{ab} = \sqrt{ab} : b$ as the formula of the "arithmetic proportion." The author may be misled by the free use of the term "proportion" among the ancients; but the "arithmetic proportion" is at any rate identical with our "arithmetic progression," wherein $a - b = b - c$.

The translation is after all the raison d'être of the dissertation. It is uneven in character, the chief fault being that Dr. Johnson follows Hoche's text too literally and thus is led into needlessly anacoluthic English (*e. g.*, Book II, Chap. 1, Section 1, and II. 27. 1) and, in a few cases, into translating a hopelessly corrupt text (*e. g.*, I. 13. 8, where ἑαυτοῦ is certainly preferable to ἑαυτῶν; II. 23. 5, where Ast's text is far better than the hopeless reading of Hoche). Somewhat more care in punctuation and in proof-reading would have saved the sense of several ambiguous passages (see, *e. g.*, I. 9. 1; and on p. 42, lines 15 ff.). But beyond this the version is not free from errors, of which the citation of a few of the most serious must suffice here. In I. 7. 4 Nicomachus states that even

numbers, when divided into two parts, show but "one of the two kinds" (τὸ ἕτερον εἶδος) of number in these parts; that is, both are odd or both even; Dr. Johnson says, "shows the first kind of number only (odd) without share in the rest." This is obviously false; because, for example, $8 = 5 + 3$ or $6 + 2$. Nicomachus tells in I. 12. 1–2 of the derivation of "composite numbers," using the word συντιθέναι in the sense of "to multiply" in the context; the translator gives the sense as "to add" and thereby creates confusion. Another serious error comes in the discussion of the "sieve of Eratosthenes" (I. 13). The Greek reads that the odd numbers "assume in turn the measuring function" (p. 33, 2, Hoche: ὥστε τὸ μὲν μετρεῖν διαδέξονται, κτλ.) in the process of discovering which terms are prime, but the translator has distorted the simple phrase into "their capacity of measuring will depend on the order in which they (*i. e.*, numbers to be measured) lie in the line" (p. 16). Still another slip gives the reader a fundamental misconception of Nicomachus's method of discovering "perfect numbers" (I. 16. 4). Dr. Johnson (p. 19) thus summarizes up the rule: "Sum up the powers of 2 and add unity to each sum; when the result is prime, multiply it by the last power of 2 in the series summed up; the product will be a perfect number." The translated section which follows is in conformity with this notion. But there is really no question of "adding unity to each sum"; εἶτα ἀεὶ κατὰ ἑνὸς πρόσθεσιν ἐπισωρεύειν (p. 41, 6, Hoche) means "then you must sum them up, as you take them one by one into the combination," as a comparison of the commentary by Iamblichus (*in Nic.*, p. 34, 2, edited by Pistelli) will amply confirm. The translator has probably failed to notice that unity is definitely included by Nicomachus in the series of (even-times even) numbers which he here uses, a fact which is made perfectly clear by the opening words of section 8 of the chapter. One leaves the reading of this dissertation with the feeling that the treatment accorded to Nicomachus is not as thorough-going nor as accurate as he deserves.

FRANK EGLESTON ROBBINS.

THE UNIVERSITY OF MICHIGAN,
DEPARTMENT OF GREEK.

PROBLEMS FOR SOLUTION.

SEND ALL COMMUNICATIONS ABOUT PROBLEMS TO B. F. FINKEL, Springfield, Mo.

ALGEBRA.

477. Proposed by J. L. RILEY, Northeastern State Normal School, Tahlequah, Oklahoma.

Evaluate the product $(1 + r + r^2 + r^3)(1 + r^3 + r^4 + r^6) \cdots (1 + r^{3^{n-1}} + r^{3^n} + r^{3 \cdot 3^{n-1}})$.

478. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

Solve the equations

$$\frac{l}{x} + \frac{y}{m} + \frac{z}{n} = 1, \qquad \frac{x}{l} + \frac{m}{y} + \frac{z}{n} = 1, \qquad \frac{x}{l} + \frac{y}{m} + \frac{n}{z} = 1.$$

GEOMETRY.

510. Proposed by JOSEPH E. ROWE, State College, Pa.

Show how to find the equation of a line parallel to a side of the triangle of reference and passing through a given point, in any system of homogeneous coördinates, using the condition that two lines are parallel in this system but not the condition that two lines are perpendicular. Illustrate the method by using it to find the trilinear coördinates of the points of contact of one escribed circle of the triangle.

511. Proposed by FRANK V. MORLEY, Student, Haverford College, Pa.

Let a_i $(i = 1, 2, 3, 4)$ be four points on a circle and let the in-center of the triangle formed by omitting a_i be c_i; prove that the four points c_i form a rectangle.

CALCULUS.

425. Proposed by O. S. ADAMS, U. S. Coast and Geodetic Survey, Washington, D. C.

Show that the infinite product

$$(1 - z)(1 + \tfrac{1}{2}z)(1 - \tfrac{1}{3}z)(1 + \tfrac{1}{4}z)\cdots = \frac{\sqrt{\pi}}{\Gamma(1 + \tfrac{1}{2}z)(\tfrac{1}{2} - \tfrac{1}{2}z)}.$$

426. Proposed by C. N. SCHMALL, New York City.

If A be the area of a plane triangle constructed with the sides a, b, c, such that

$$a^3 + b^3 + c^3 = 3k^3,$$

show that the maximum value of A is $\frac{1}{4}k^2$.

MECHANICS.

342. Proposed by WILLIAM HOOVER, Columbus, Ohio.

A uniform rod of length $2a$ is freely hinged at one end, at the other end a string of length b is attached which is fastened at its further end to a point on the surface of a homogeneous sphere of radius c. If the masses of the rod and sphere are equal, find the motion of the system when slightly disturbed from the vertical, and the cubic equation giving the corresponding small oscillations.

343. Proposed by J. ROSENBAUM, New Haven, Conn.

Two bodies of equal masses and coefficients of friction μ_1 and μ_2 are connected by a light spring of stiffness k and placed on an inclined plane. Discuss the motion of each body when the angle between the non-stretched spring and the plane is θ.

NUMBER THEORY.

261. Proposed by NORMAN ANNING, Chilliwack, B. C.

Show that for any positive integer n (excluding powers of 2) positive integers a_1, a_2, a_3, $\cdots$ a_k, which are less than $n/2$ can be chosen in such a way that

$$2^k \cos(a_1\pi/n) \cos(a_2\pi/n) \cos(a_3\pi/n) \cdots \cos(a_k\pi/n) = 1.$$

262. Proposed by C. N. SCHMALL, New York City.

If x, y, z are three integers, consecutive among the integers prime to 3, show that

$$x(x - 2y) - z(z - 2y) = \pm 3.$$

SOLUTIONS OF PROBLEMS.

ALGEBRA.

J. A. BULLARD and J. W. BALDWIN solved 464. These solutions were received after selections for publication were made.

465. Proposed by CYRUS B. HALDEMAN, Ross, Ohio.

Having given $\tan^{-1} 1 = \tan^{-1} 1/2 + \tan^{-1} 1/3$, show that

$$\tan^{-1} 1 = 5 \tan^{-1} 1/8 + 2 \tan^{-1} 1/18 + 3 \tan^{-1} 1/57.$$

SOLUTION BY HORACE OLSON, Chicago, Illinois.

Put $\tan^{-1} 1/2 = \tan^{-1} 1/3 + \tan^{-1} x_1 = \tan^{-1} \dfrac{3x_1 + 1}{3 - x_1}$. From this equation x_1 is found to be 1/7. Hence, $\tan^{-1} 1 = 2 \tan^{-1} 1/3 + \tan^{-1} 1/7$.

Now put

$$\tan^{-1} 1/3 = \tan^{-1} 1/7 + \tan^{-1} x_2 = \tan^{-1} \frac{7x_2 + 1}{7 - x_2};$$

whence $x_2 = 2/11$, and $\tan^{-1} 1 = 3 \tan^{-1} 1/7 + 2 \tan^{-1} 2/11$.

Now put

$$\tan^{-1} 1/7 = \tan^{-1} x_3 + \tan^{-1} y_3 = \tan^{-1} \frac{x_3 + y_3}{1 - x_3 y_3}.$$

This gives the indeterminate equation $x_3 y_3 + 7(x_3 + y_3) = 1$, or $(x_3 + 7)(y_3 + 7) = 50$. One set of solutions of this equation is $x_3 = 1/8$, $y_3 = 1/57$. Hence, $\tan^{-1} 1 = 3 \tan^{-1} 1/8 + 3 \tan^{-1} 1/57 + 2 \tan^{-1} 2/11$. Put

$$\tan^{-1} 2/11 = \tan^{-1} 1/8 + \tan^{-1} x_4 = \tan^{-1} \frac{8x_4 + 1}{8 - x_4};$$

whence $x_4 = 1/18$. Hence, $\tan^{-1} 1 = 5 \tan^{-1} 1/8 + 2 \tan^{-1} 1/18 + 3 \tan^{-1} 1/57$.

Also solved by E. B. ESCOTT, W. J. THOME, M. T. REED, G. W. HARTWELL, and E. E. WHITEFORD.

466. Proposed by E. B. ESCOTT, Kansas City, Mo.

For what functions, f, are the following relations true:

When

$$\frac{f(x, y, z)}{X} = \frac{f(y, z, x)}{Y} = \frac{f(z, x, y)}{Z},$$

then

$$\frac{f(X, Y, Z)}{x} = \frac{f(Y, Z, X)}{y} = \frac{f(Z, X, Y)}{z}?$$

SOLUTION BY ALBERT A. BENNETT, University of Texas.

We notice that $f(X, Y, Z)$ must be homogeneous, since if we replace X, Y, Z by cX, cY, cZ, in the first relation it is unaltered, and hence also the second. Thus x, y, z may be regarded as the homogeneous coördinates of the points in one plane, X, Y, Z of those in a second. The problem may therefore be expressed as follows: What are the involutoric plane transformations with triangular symmetry? Here "involutoric" is used in the restricted sense as of period two, and triangular symmetry is used in approximately the sense first suggested by Clifford, the point (1, 1, 1) being a center of triangular symmetry. There can be little doubt that there exist transformations of this form which are essentially transcendental and which may be regarded as limiting cases of algebraic birational transformations. To catalogue the explicit forms of even the algebraic cases is perhaps out of the question, since no explicit classification of algebraic forms of Cremona transformations has been attempted beyond the very simplest cases. Normal forms under the Cremona group are indeed known. Compare Pascal's *Repertorium* or other detailed articles on birational geometry.

It is furthermore obvious that a given geometric solution gives rise to an infinite number of analytic solutions. For example, if $g(x, y, z)$ be one solution, and $\varphi(x, y, z)$ be a constant or the equation of any triangularly symmetric self-corresponding curve whether the self-correspondence be singular or not, then $\varphi(x, y, z)g(x, y, z)$ is a solution. Apart from the trivial solutions $f(x, y, z) = 0$ and $f(x, y, z) = x$, the simplest case is probably the quadratic transformation given by $f(x, y, z) = 1/x$, and then taking $\varphi(x, y, z) = xyz$, or $\sqrt{xyz}$, or $xyz/\log(1 + xyz)$ or $(xy + yz + zx)(x + y + z)$, etc., we get other related solutions from $f(x, y, z) = 1/x$.

GEOMETRY.

An excellent solution of 485 by Analytical Geometry was received from O. S. ADAMS after selections for publication were made.

493. Proposed by FLORENCE P. LEWIS, Goucher College, Baltimore, Md.

Construct three circles each of which shall be tangent to the other two and to two sides of a given triangle.

SOLUTION BY C. F. GUMMER, Queen's University, Kingston, Ont.

If we impose the following conditions: (i) each circle is to touch a different pair of sides; (ii) the circles must be real; (iii) no two circles may coincide; we can get 68 solutions.

With the extra condition (iv) all points of contact must be distinct, we get only 32 solutions. We shall obtain formulae for these 32 cases, as being the most interesting.

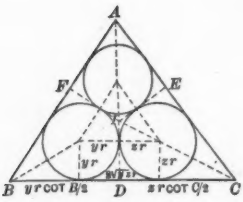


Figure for Equation 6.

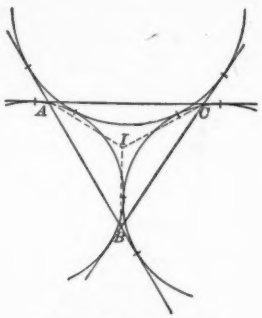


Figure for Equation 7.

Let ABC be the triangle, I, D, E, F the center and points of contact of the in-circle, I_1, D_1, $\cdots$ those of the ex-circles, r, r_1, r_2, r_3 the radii of these circles.

The above conditions show that the required circles must touch externally. Consider any three circles touching externally. If we draw a common tangent to each pair, observing (iv), we get one of eight possible triangles, and in every case the centers of the circles lie on three *concurrent* bisectors of the angles. Hence, our solutions will appear in four groups according as we place the centers on the set of bisectors meeting in I, I_1, I_2, or I_3.

Take the case where the centers are to lie on AI, BI, CI. Let the radii of the required circles be xr, yr, zr, positive in the direction of r. The conditions that the cricles touch one another externally are (since $a/r = \cot B/2 + \cot C/2$)

$$y \cot B/2 + z \cot C/2 \pm 2\sqrt{yz} = \cot B/2 + \cot C/2, \tag{1}$$

$$z \cot C/2 + x \cot A/2 \pm 2\sqrt{zx} = \cot C/2 + \cot A/2, \tag{2}$$

and

$$x \cot A/2 + y \cot B/2 \pm 2\sqrt{zx} = \cot A/2 + \cot B/2, \tag{3}$$

from which $\sqrt{x}$, $\sqrt{y}$, $\sqrt{z}$ may be found. Since by reversing two of the ambiguities we only change the sign of the square root of one of the unknowns, they furnish not eight but two cases as regards x, y, and z, according as an odd or even number of them are minus signs. We may therefore take them all plus or all minus. To solve the equations, eliminate the constant terms between (1) and (2), and also between (1) and (3), obtaining

$$(\sqrt{x} \cos A/2 \pm \sqrt{z} \sin A/2)^2 = (\sqrt{y} \cos B/2 \pm \sqrt{z} \sin B/2)^2 \tag{4}$$

and

$$(\sqrt{x} \cos A/2 \pm \sqrt{y} \sin A/2)^2 = (\sqrt{z} \cos C/2 \pm \sqrt{y} \sin C/2)^2, \tag{5}$$

which give the ratios of x, y, and z for the 8 cases corresponding to the set of internal bisectors. If we take the positive square roots in (4) and (5), we find, using the plus signs,

$$x : y : z = (1+u)^{-2} : (1+v)^{-2} : (1+w)^{-2} \tag{6}$$

and, using the minus signs,

$$x : y : z = u^2(1+u)^{-2} : v^2(1+v)^{-2} : w^2(1+w)^{-2} \tag{7}$$

where

$$u = \tan A/4, \qquad v = \tan B/4, \qquad w = \tan C/4;$$

and hence

$$1 - \Sigma u - \Sigma uv + uvw = 0. \tag{8}$$

By (1), the actual values of x, y, and z are $\dfrac{(1+v)(1+w)}{2(1+u)}$, etc., in the case of (6) and $\dfrac{u(1+v)(1+w)}{2vw(1+u)}$, etc., in the case of (7). From these two solutions we can get, not only the 8, but all the 32, by replacing A, B, C by $A + l\pi$, $B + m\pi$, $C + n\pi$, where $l + m + n$ is a multiple of 4, since these angles apply equally well to the triangle, and they leave the relation (8) unaltered.

To get a practical construction, let us denote by $\rho_1, \rho_{11}, \rho_{12}, \rho_{13}; \rho_2, \rho_{21}, \rho_{22}, \rho_{23}, \cdots$, the in- and ex-radii of AFI, BDI, CEI. Then

$$\rho_{11} = \frac{1+u}{2}\,r, \qquad \rho_{12} = \frac{1+u}{2u}\,r, \text{ etc.}$$

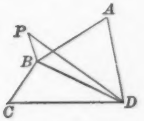
Hence in the case of (6), the radii of the required circles are the three fourth proportionals to ρ_{11}, ρ_{21}, ρ_{31} in different orders; and in the case of (7) the fourth proportionals to ρ_{12}, ρ_{22}, ρ_{32}. The remainder of the set of 8 cases may be solved by using other combinations of the ρ's, while all the 32 cases may be similarly treated by means of the triangles AF_1I_1, etc.

495. Proposed by N. P. PANDYA, Sojitra, India.

A point P moves so that the quadrilateral $PBCD$ is half of a given quadrilateral $ABCD$. Find the locus of P.

SOLUTION BY J. W. BALDWIN, University of Michigan.

In general the triangles ABD and BCD will not be equal. Let ABD be the larger of the two. Then we are to have triangle BCD + triangle PBD equal to half of the given quadrilateral $ABCD$ for all positions of P. That is, the triangle PBD must have a constant area; and having a fixed base BD must have a constant altitude, the distance from P to BD or BD produced. Hence the locus of P is a line parallel to BD. In case triangle ABD = triangle BCD the locus of P is the line of which BD is a segment and two sides PB and PD of the quadrilateral $PBCD$ fall in this line.



Also solved by W. J. THOME, G. W. HARTWELL, WILLIAM HOOVER, S. W. REAVES, W. R. RANSOM, J. W. CLAWSON, and the PROPOSER, some solvers using analytic methods and one using trilinear, perpendicular coördinates.

CALCULUS.

407. Proposed by PAUL CAPRON, Annapolis, Maryland.

A coffee pot in the form of a conical frustum, 10 inches high, with a lower base 8 inches in diameter and an upper base 6 inches in diameter, is held on a slant so that the lower base is barely covered by the coffee within, and the upper base is barely uncovered. How much coffee does the pot contain?

III. SOLUTION BY WILLIAM W. JOHNSON, Cleveland, Ohio.

The quantity of coffee in the pot is equal to the volume of the conical ungula C-$APBQ$ formed by tipping the conical frustum on a slant, according to the conditions of the problem. In the figure let $AL = R$, $DF = r$, $FL = h$, $GO = x$, $FO = y$. Then,

Volume $C\text{-}APBQ = V = \int_r^R S dy = \frac{h}{R-r}\int_r^R \left(x^2 \arccos \frac{2Rr-(R+r)x}{(R-r)x} \right.$

$$\left. - \frac{1}{(R-r)^2}[2Rr-(R+r)x]\sqrt{4Rr(R+r)x - 4Rrx^2 - 4R^2r^2} \right) dx,$$

in which, area $HKE = S$, and

$$dy = \left(\frac{h}{R-r}\right) dx;$$

since

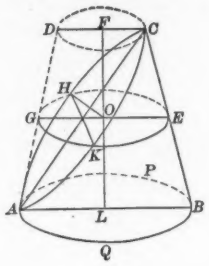
$$y = \frac{h(x-r)}{R-r}.$$

Integrating, we obtain

$$V = \frac{\pi h R^3}{3(R-r)}(R^3 - r^3).^1$$

Putting $h = 10$, $R = 4$, and $r = 3$, we get

$$V = \frac{80}{3}(8 - 3\sqrt{3})\pi = 234.895 \text{ cu. in.}$$



409. Proposed by B. J. BROWN, Victor, Colorado.

Integrate the equation

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{1}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) - \frac{2}{(x+y)^2} z = 0.$$

SOLUTION BY O. S. ADAMS, Coast and Geodetic Survey, Washington, D. C.

Let

$$z = \frac{u}{(x+y)^2}. \tag{1}$$

Then, computing $\partial z/\partial x$, $\partial z/\partial y$ and $\partial^2 z/\partial x \partial y$, and substituting in the given equation, we may write the result as follows,

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} - \frac{u}{x+y}\right) - \frac{1}{x+y}\left(\frac{\partial u}{\partial x} - \frac{u}{x+y}\right) - \frac{2u}{(x+y)^2} = 0.$$

Now let

$$\frac{\partial u}{\partial x} - \frac{u}{x+y} = v. \tag{2}$$

Then

$$u = \frac{(x+y)^2}{2}\frac{\partial v}{\partial y} - \frac{x+y}{2}v. \tag{3}$$

Computing $\partial u/\partial x$ and $u/(x+y)$, and substituting in (2), we have

$$\frac{\partial u}{\partial x} - \frac{u}{x+y} = \frac{(x+y)^2}{2}\frac{\partial^2 v}{\partial x \partial y} + \frac{x+y}{2}\frac{\partial v}{\partial y} - \frac{x+y}{2}\frac{\partial v}{\partial x} = v;$$

or

$$\frac{\partial^2 v}{\partial x \partial y} + \frac{1}{x+y}\frac{\partial v}{\partial y} - \frac{1}{x+y}\frac{\partial v}{\partial x} - \frac{2v}{(x+y)^2} = 0.$$

This equation may be written in the form,

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{v}{x+y}\right) - \frac{1}{x+y}\left(\frac{\partial v}{\partial x} + \frac{v}{x+y}\right) = 0.$$

Finally, let

$$\frac{\partial v}{\partial x} + \frac{v}{x+y} = w. \tag{4}$$

Then

$$\frac{\partial w}{\partial y} - \frac{w}{x+y} = 0; \quad \text{or} \quad \frac{1}{x+y}\frac{\partial w}{\partial y} - \frac{w}{(x+y)^2} = 0,$$

[1] See Finkel's *Solution Book*, p. 319.

which may be written

$$\frac{\partial}{\partial y}\left(\frac{w}{x+y}\right) = 0, \text{ or on integrating, } \frac{w}{x+y} = \phi'''(x),$$

$\phi'''(x)$ being an arbitrary function of x.

Substituting the value of w thus given in (4), we find

$$(x+y)\frac{\partial v}{\partial x} + v = (x+y)^2\phi'''(x), \quad \text{or} \quad \frac{\partial}{\partial x}[(x+y)v] = (x+y)^2\phi'''(x).$$

Integrating with respect to x, the right-hand member being integrated twice by parts, we find

$$(5) \qquad (x+y)v = (x+y)^2\phi''(x) - 2(x+y)\phi'(x) + 2\phi(x) + 2\psi(y),$$

$\psi(y)$ being an arbitrary function of y with the factor 2 included for symmetry.

From (3), we have

$$(x+y)\frac{\partial v}{\partial y} - v = \frac{2u}{x+y}.$$

If we differentiate (5) with respect to y, we obtain

$$(x+y)\frac{\partial v}{\partial y} + v = 2(x+y)\phi''(x) - 2\phi'(x) + 2\psi'(y).$$

By substituting

$$(x+y)\frac{\partial v}{\partial y} = v + \frac{2u}{x+y},$$

we find

$$2v + \frac{2u}{x+y} = 2(x+y)\phi''(x) - 2\phi'(x) + 2\psi'(y);$$

or

$$(x+y)v + u = (x+y)^2\phi''(x) - (x+y)\phi'(x) + (x+y)\psi'(y).$$

Substituting the value of $(x+y)v$ from (5), we obtain

$$u = (x+y)[\phi'(x) + \psi'(y)] - 2\phi(x) - 2\psi(y).$$

Hence, from (1),

$$z = \frac{\phi'(x) + \psi'(y)}{x+y} - \frac{2\phi(x) + 2\psi(y)}{(x+y)^2};$$

or

$$z = (x+y)\left[\frac{\partial}{\partial x}\frac{\phi(x)}{(x+y)^2} + \frac{\partial}{\partial y}\frac{\psi(y)}{(x+y)^2}\right].$$

412. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

Given a triangular field of sides a, b, and c. Show how to divide the field into two equal parts by a straight fence so that the cost of the fence is the least.

SOLUTION BY W. R. RANSOM, Tufts College.

Suppose $a > b > c$. Connect points on the sides a and b so as to form a new triangle with the same included angle and sides of length xa and yb. If this triangle has half the area of the given triangle, $y = 1/2x$. The square of length of the fence is then

$$\varphi(x) = a^2x^2 + \frac{b^2}{4x^2} - (a^2 + b^2 - c^2).$$

The minimum value of $\phi(x)$ is found to be $\frac{1}{2}[c^2 - (a-b)^2]$, where $x = \sqrt{b/2a}$ and $y = \sqrt{a/2b}$. From $a < b + c$ and $c < b$ we get $a < 2b$; hence $y < 1$. Similarly, $x < 1$ so that this fence lies wholly within the given field. Moreover, this fence is less than the minimum fence built across either of the other corners; for, multiply $a < b + c$ by $2(b - c)$ which is positive, and add $c^2 - b^2$, then, $c^2 - b^2 + 2ab - 2ac < b^2 - c^2$, or $c^2 - (b-a)^2 < b^2 - (c-a)^2$. Similarly, interchanging a and b, $c^2 - (b-a)^2 < a^2 - (c-b)^2$. This minimum triangle is isosceles, the sides along a and b being each $= \sqrt{ab}/2$.

Also solved by O. S. ADAMS, C. N. SCHMALL, PAUL CAPRON, and HORACE OLSON.

MECHANICS.

324. Proposed by H. S. UHLER, Yale University.

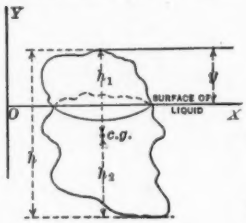
A rigid body of any shape is at rest in a neutral liquid which is also at rest and has an indefinitely great volume. The body is so situated that the free surface of the liquid is tangent to it at its highest point (or points). All the space above the liquid is filled with a neutral stagnant fluid whose density is not greater than the density of the liquid. Show that the work done in raising (pure translation) the body very slowly until the interface of the two fluids is tangent to it at its lowest point (or points) is expressible by the formula $mgh - gV(\rho_1 h_1 + \rho_2 h_2)$, where $m \equiv$ mass of body, $V \equiv$ volume of body, $\rho_1 \equiv$ mean density of lower medium, $\rho_2 \equiv$ density of upper medium, $h_1 \equiv$ distance of center of mass of the displaced liquid below the free surface in the initial position of the body, $h_2 \equiv$ elevation of center of mass of displaced fluid above the interface in final position of the body and $h = h_1 + h_2$. (Neglect surface-tension, etc.)

SOLUTION BY J. B. EPPES, Annapolis, Md.

Take the axes indicated in the figure. Let the highest point of the body at any time be a distance y above the surface of the liquid, and let A be area of the cross-section of the body made by the surface of the liquid.

Then the mass of the fluid displaced is $\rho_2 \int_0^y A dy$ and the mass of the liquid displaced is $\rho_1 \left(V - \int_0^y A dy \right)$.

Hence, the downward force is $g \left[m - \rho_2 \int_0^y A dy - \rho_1 \left(V - \int_0^y A dy \right) \right]$. Then, the total work in raising the body, is



$$\int_0^h \left[mg - \rho_2 g \int_0^y A dy - \rho_1 g \left(V - \int_0^y A dy \right) \right] dy = mgh$$

$$- \rho_2 g \int_0^h dy \int_0^y A dy - \rho_1 g V h + \rho_1 g \int_0^h dy \int_0^y A dy.$$

Now assume A as a function of y to be of the form

$$A \equiv B + Cy + Dy^2 + \cdots.$$

Then,

$$\int_0^h dy \int_0^y A dy = \int_0^h \left(By + \frac{Cy^2}{2} + \frac{Dy^3}{3} + \cdots \right) dy = \frac{Bh^2}{2} + \frac{Ch^3}{2 \cdot 3} + \frac{Dh^4}{3 \cdot 4} + \cdots$$

$$= \left[Bh^2 + \frac{Ch^3}{2} + \frac{Dh^4}{3} + \cdots \right] - \left[\frac{Bh^2}{2} + \frac{Ch^3}{3} + \frac{Dh^4}{4} + \cdots \right]$$

$$= h \int_0^h A dy - \int_0^h y A dy = hV - Vh_1 = Vh_2.$$

Hence, work $= mgh - \rho_2 g V h_2 - \rho_1 g V h + \rho_1 g V h_2 = mgh - gV(\rho_1 h_1 + \rho_2 h_2)$.

325. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

The lever of a testing machine is c feet long, and is poised on a knife edge a feet from one end and b feet from the other, and in a horizontal line above which the beam is symmetrical. The beam is m inches deep at the knife edge and tapers uniformly to a depth of n inches at each end; the width of the beam is the same throughout its length. Find the distance of the center of gravity of the beam from the knife edge.

Solution by W. J. Thome, University of Detroit.

It is a well known fact that if the parallel sides of a trapezoid are g and s, and the perpendicular distance between them is p, then the distance from the greater side g to the center of gravity of the trapezoid is

$$\frac{p}{3}\frac{(g+2s)}{(g+s)}.$$

Let t be the constant width of the lever, in inches;
d, the density of the material of the lever,
M, the total mass of the lever;
m_1, the mass of that part of the lever from the knife edge to the end distant a feet away;
m_2, the mass of that part of the lever from the knife edge to the end distant b feet away;
$\overline{X}$, the distance in feet from the knife edge to the center of gravity of the entire lever;
$\bar{x}_1$, the distance in feet from the knife edge to the center of gravity of the mass m_1; and
x_2, the distance in feet from the knife edge to the center of gravity of the mass m_2.
Then, using the foot as our unit of length, we have

$$M\overline{X} = m_2\bar{x} - m_1\bar{x}_1;$$

or

$$\left[\frac{1}{2}\left(\frac{m+n}{12}\right)a\frac{t}{12}d+\frac{1}{2}\left(\frac{m+n}{12}\right)b\frac{t}{12}d\right]\overline{X}$$

$$=\left[\frac{1}{2}\left(\frac{m+n}{12}\right)b\frac{t}{12}d\right]\left[\frac{b\left(\frac{m+2n}{12}\right)}{3\left(\frac{m+n}{12}\right)}\right]-\left[\frac{1}{2}\left(\frac{m+n}{12}\right)a\frac{t}{12}d\right]\left[\frac{a\left(\frac{m+2n}{12}\right)}{3\left(\frac{m+n}{12}\right)}\right].$$

Hence,

$$\overline{X} = \frac{1}{3}\frac{(m+2n)}{(m+n)}(b-a),$$

which is to be measured in the direction of whichever is the greater, b or a.

NUMBER THEORY.

240. Proposed by J. W. NICHOLSON, Louisiana State University.

If the roots of $x^3 - px + q = 0$ are rational, prove that $4p - 3x^2$ is a perfect square.

Solution by William E. Patten, Government Institute of Technology, Shanghai, China.

Let the roots of $x^3 - px + q = 0$ be x_1, x_2, x_3. Then

$$x_1 + x_2 + x_3 = 0;$$

and

$$x_1x_2 + x_2x_3 + x_3x_1 = -p.$$

Therefore,

$$\begin{aligned} 4p - 3x_1^2 &= -4(x_1x_2 + x_2x_3 + x_3x_1) - 3x_1^2 \\ &= -4x_2x_3 - 4x_1(x_2 + x_3) - 3x_1^2 \\ &= -4x_2x_3 - 4(-x_2 - x_3)(x_2 + x_3) - 3(-x_2 - x_3)^2 \\ &= -4x_2x_3 + (x_2 + x_3)^2 = (x_2 - x_3)^2. \end{aligned}$$

Since the coefficient of the leading term of the given equation is unity, and the roots are rational, they are also integral.

Therefore, $x_2 - x_3$ is an integer, and $4p - 3x_1^2$ is a perfect square.

Similarly for the other roots:

$$4p - 3x_2^2 = (x_3 - x_1)^2, \quad \text{and} \quad 4p - 3x_3^2 = (x_1 - x_2)^2.$$

Also solved by H. N. CARLETON, J. E. ROWE, L. C. MATHEWSON, A. H. HOLMES, ELIJAH SWIFT, O. S. ADAMS, J. ROSENBAUM, LOUIS CLARK, E. E. WHITEFORD, and H. S. UHLER.

241. Proposed by CLIFFORD N. MILLS, Brookings, S. Dak.

If $a^2 + b^2 = c^2$, where a, b, and c are integers, then prove that abc will be a multiple of 60.

SOLUTION BY ALBERT G. CARIS, Defiance College.

From the well-known theorem that any integral solution of $a^2 + b^2 = c^2$ may be put in the form $2xy$, $x^2 - y^2$, $x^2 + y^2$, where x and y are integers, it follows immediately that

$$abc = 2xy(x - y)(x + y)(x^2 + y^2).$$

Showing that this product is always a multiple of 3, 4, and 5 is sufficient to prove the proposed problem.

I. We may write $x = 2m - 1$, or $2m$ and $y = 2n - 1$, or $2n$, where m and n are integers. Whenever $x = 2m$, or $y = 2n$, abc is a multiple of 4. In all other cases $x = 2m - 1$ at the same time that $y = 2n - 1$ and consequently, $x - y$, $x + y$, and $x^2 + y^2$ are all multiples of 2.

Therefore abc is always a multiple of 4.

II. We may write $x = 3r - 1$, $3r$, or $3r + 1$ and $y = 3s - 1$, $3s$, or $3s + 1$, where r and s are integers. Whenever $x = 3r$, or $y = 3s$, abc is a multiple of 3. The combinations resulting from all other cases may be arranged in the two groups below:

Group A		*Group B*	
x	y	x	y
$3r - 1$	$3s - 1$	$3r - 1$	$3s + 1$
$3r + 1$	$3s + 1$	$3r + 1$	$3s - 1$

From combinations of group A the factor $x - y = 3(r - s)$. From combinations of group B the factor $x + y = 3(r + s)$. Therefore abc is always a multiple of 3.

III. We may write

$$x = 5u - 2, \quad 5u - 1, \quad 5u, \quad 5u + 1, \quad \text{or} \quad 5u + 2$$

and

$$y = 5v - 2, \quad 5v - 1, \quad 5v, \quad 5v + 1, \quad \text{or} \quad 5v + 2,$$

where u and v are integers. Whenever $x = 5u$, or $y = 5v$, abc is a multiple of 5. The combinations resulting from all other cases may be arranged in the three groups below:

Group C		*Group D*		*Group E*	
x	y	x	y	x	y
$5u - 2$	$5v - 2$	$5u - 2$	$5v + 2$	$5u \pm 2$	$5v \pm 1$
$5u - 1$	$5v - 1$	$5u - 1$	$5v + 1$		
$5u + 1$	$5v + 1$	$5u + 1$	$5v - 1$	$5u \pm 1$	$5v \pm 2$
$5u + 2$	$5v + 2$	$5u + 2$	$5v + 2$		

All combinations of group C make $x - y = 5(u - v)$. All combinations of group D make $x + y = 5(u + v)$. All combinations of group E make $x^2 + y^2 = 5(5u^2 \pm 4u + 5v^2 \pm 2v + 1)$ or $5(5u^2 \pm 2u + 5v^2 \pm 4v + 1)$. Therefore, abc is always a multiple of 5. Hence, abc is always divisible by 60.

Also solved by S. A. COREY, A. H. HOLMES, HORACE OLSON, J. W. CLAWSON, J. ROSENBAUM, J. E. ROWE, H. C. FEEMSTER, H. N. CARLETON, W. J. THOME, ELIJAH SWIFT, and E. E. WHITEFORD.

243. Proposed by CLIFFORD N. MILLS, Brookings, South Dakota.

Determine rational values of x that will render $x^3 + px^2 + qx + r$ a perfect cube. Apply the result to $x^3 - 8x^2 + 12x - 6$.

Solution by E. B. Escott, Kansas City, Mo.

Let $x^3 + px^2 + qx + r = (x + a)^3$. Expanding and collecting terms

$$(3a - p)x^2 + (3a^2 - q)x + (a^3 - r) = 0. \tag{1}$$

If x is rational, the discriminant must be a square, that is,

$$(3a^2 - q)^2 - 4(3a - p)(a^3 - r) = d^2. \tag{2}$$

One simple solution is $a = p/3$.

Then from (1), we get

$$x = -\frac{a^3 - r}{3a^2 - q} = -\frac{\frac{p^3}{27} - r}{\frac{p^2}{3} - q} = -\frac{p^3 - 27r}{9(p^2 - 3q)}. \tag{3}$$

Other solutions of (2) may be found by Euler's method. Expanding (2), we have

$$-3a^4 + 4pa^3 - 6qa^2 + 12ra + (q^2 - 4pr) = d^2.$$

If we know one solution, we can usually find as many as desired.

Applying the above results to $x^3 - 8x^2 + 12x - 6$, we have $p = -8$, $q = 12$, $r = -6$. By (3), we have $x = 25/18$. (2) becomes $(3a^2 - 12)^2 - 4(3a + 8)(a^3 + 6) = d^2$; or, expanded,

$$-3a^4 - 32a^3 - 72a^2 - 72a - 48 = d^2.$$

$a = -2$ is a solution, which gives $x = \pm 1$. Let $a = b - 2$. Then,

$$-3b^4 - 8b^3 + 48b^2 - 72b + 16 = d^2 = (kb^2 + lb + m)^2.$$

Expanding,

$$(k^2 + 3)b^4 + 2(kl + 4)b^3 + (2km + l^2 - 48)b^2 + 2(lm + 36)b + (m^2 - 16) = 0.$$

Let $m = 4$, $l = -9$, $k = -33/8$. Then

$$b = -\frac{2(kl + 4)}{k^2 + 3} = -\frac{752}{183} \quad \text{and} \quad a = -\frac{1118}{183}.$$

Substituting in (1), we have

$$x = \frac{51287}{8235} \quad \text{and} \quad \frac{1895}{549}.$$

Another value of a is -8, whence $x = 23/4$ and $11/2$. Also

$$a = -\frac{632}{361}, \quad -\frac{74}{13}, \quad -\frac{5738}{1381}, \quad \ldots.$$

The corresponding values of x are easily found.

Also solved by Elijah Swift, Norman Anning, J. A. Colson, and J. E. Rowe.

Editorial Note.—The problem in effect is to find rational points on the cubic

$$x^3 + px^2 + qx + r - y^3 = 0.$$

If the discriminant of $x^3 + px^2 + qx + r$ vanishes this cubic is rational and an infinity of rational points may be found by the method of section by a line through the double point. If not it is a cubic of one branch and genus 1. The theory of the rational points has been discussed by H. Poincaré, *Liouville Journal*, 1901, 161.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

NEW QUESTION.

34. Given the mixed integral and functional equation

$$\int_{x=0}^{x=x} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{x}{2}\right) + f(x)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

DISCUSSIONS.

I. CONCERNING AN ILLUSTRATION OF A CERTAIN NECESSARY CONDITION IN MINIMIZING A DEFINITE INTEGRAL WITH DISCONTINUOUS INTEGRAND.

By PAUL R. RIDER, Washington University.

The problem of minimizing a definite integral in which the integrand possesses a finite discontinuity along a plane curve has been considered by Bliss and Mason.[1] A certain necessary condition that they have discovered may be stated as follows:

If a curve C, of parameter t, which passes from the fixed point P_0 to the fixed point P_2 and crosses a curve D, whose equations are $x = x(\alpha)$, $y = y(\alpha)$, minimizes the sum of the two integrals

$$I = \int_{t_0}^{t_1} F(x, y, x', y')dt, \qquad i = \int_{t_1}^{t_2} f(x, y, x', y')dt,$$

the first integral to be taken from the point P_0 to the curve D and the second from the curve D to the point P_2, then at P_1, the point of intersection of C and D, the relation

$$x_a(F_{x'} - f_{x'}) + y_a(F_{y'} - f_{y'}) = 0$$

must hold. The argument of x_a, y_a is the value of α for the point P_1; the arguments of $F_{x'}$, $F_{y'}$ are the values of x, y, x', y' on the curve C_{01} at the point P_1, and those of $f_{x'}$, $f_{y'}$ are the values of the same variables on the curve C_{12} at the point P_1.

This condition is well illustrated in a problem (Calculus 389) which recently appeared in the MONTHLY:

> A man is at the southeast corner of a section of land and wishes to walk to the opposite corner in the least possible time. A circular track with a radius of $1/\pi$ miles is located in the section tangent to the west line at a point 120 rods from the south line. Conditions are such that he can walk at the rate of 4 miles an hour inside the track and 3 miles an hour outside the track. What course should he choose and how long is it? See the figure.

Solutions of the problem were given in the MONTHLY, Vol. 23, No. 24 (April, 1916), p. 125 by H. S. Uhler, and p. 127 by A. H. Holmes.

[1] BLISS and MASON, "A Problem of the Calculus of Variations in which the Integrand is Discontinuous," *Transactions of the American Mathematical Society*, Vol. 7 (1906), pp. 325–336.

Since the time is to be the least possible, the sum of the three integrals

$$I_{01} = \int_{t_0}^{t_1} \frac{1}{3}\sqrt{x'^2 + y'^2}\,dt, \qquad i = \int_{t_1}^{t_2} \frac{1}{4}\sqrt{x'^2 + y'^2}\,dt, \qquad I_{23} = \int_{t_2}^{t_3} \frac{1}{3}\sqrt{x'^2 + y'^2}\,dt$$

must be minimized.

Here the curve D is the circle, the joint P_0 being the origin,

$$x = \frac{1}{\pi}\cos\alpha - 1 + \frac{1}{\pi}, \qquad y = \frac{1}{\pi}\sin\alpha + \frac{3}{8},$$

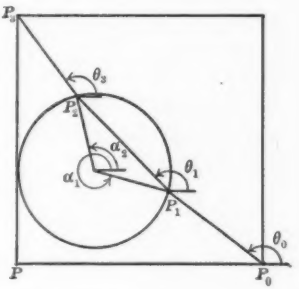
and the curve C is the broken line $P_0P_1P_2P_3$. (See figure.) There are two points of discontinuity, P_1 and P_2. For simplicity let us set

$$F = \tfrac{1}{3}\sqrt{x'^2 + y'^2}, \qquad f = \tfrac{1}{4}\sqrt{x'^2 + y'^2}.$$

It follows that the equations

$$(1)\qquad \begin{aligned} &x_a(F_{x'} - f_{x'}) + y_a(F_{y'} - f_{y'}) = 0,\\ &x_a(f_{x'} - F_{x'}) + y_a(f_{y'} - F_{y'}) = 0 \end{aligned}$$

must be satisfied at P_1 and P_2 respectively. But



$$x_a = -\frac{1}{\pi}\sin\alpha, \qquad y_a = \frac{1}{\pi}\cos\alpha,$$

$$(2)\qquad F_{x'} = \frac{x'}{3\sqrt{x'^2 + y'^2}}, \qquad F_{y'} = \frac{y'}{3\sqrt{x'^2 + y'^2}}.$$

$$f_{x'} = \frac{x'}{4\sqrt{x'^2 + y'^2}}, \qquad f_{y'} = \frac{y'}{4\sqrt{x'^2 + y'^2}}.$$

If θ is the angle that a given line makes with the positive x-axis (taking PP_0 as the positive x-direction), then

$$F_{x'} = \tfrac{1}{3}\cos\theta, \qquad F_{y'} = \tfrac{1}{3}\sin\theta, \qquad f_{x'} = \tfrac{1}{4}\cos\theta, \qquad f_{y'} = \tfrac{1}{4}\sin\theta.$$

If we use these values, and those of x_a, y_a taken from (2), equations (1) assume the form

$$(3)\qquad \begin{aligned} &-\sin\alpha_1(\tfrac{1}{3}\cos\theta_0 - \tfrac{1}{4}\cos\theta_1) + \cos\alpha_1(\tfrac{1}{3}\sin\theta_0 - \tfrac{1}{4}\sin\theta_1) = 0,\\ &-\sin\alpha_2(\tfrac{1}{4}\cos\theta_1 - \tfrac{1}{3}\cos\theta_2) + \cos\alpha_2(\tfrac{1}{4}\sin\theta_1 - \tfrac{1}{3}\sin\theta_2) = 0. \end{aligned}$$

The meanings of α_1, α_2, θ_0, θ_1, θ_2 are evident from the accompanying figure.

From the solutions of the problem by Professor Uhler and Mr. Holmes are obtained the values

$$\alpha_1 = 360° - 15° 45' 25'', \qquad \theta_0 = 90° + 52° 45' 48'',$$
$$\alpha_2 = 180° - 75° 6' 19'', \qquad \theta_1 = 90° + 44° 5' 13'',$$
$$\theta_2 = 90° + 37° 1' 51''.$$

Whence,

$$\begin{array}{llllll} & \sin\alpha_1 = -.271557, & \cos\alpha_1 = .962422, & \sin\theta_0 = .605109, & \cos\theta_0 = -.796143, \\ (4) & \sin\alpha_2 = .966400, & \cos\alpha_2 = -.257044, & \sin\theta_1 = .718285, & \cos\theta_1 = -.695749, \\ & & & \sin\theta_2 = .798312, & \cos\theta_2 = -.602245. \end{array}$$

If we substitute these numerical values in the first equation of (3) the left side becomes

$$.271557(-\tfrac{1}{2} \times .796143 + \tfrac{1}{2} \times .695749) + .962422(\tfrac{1}{2} \times .605109 - \tfrac{1}{2} \times .718285) = .003532.$$

Substituting the values (4) in the second equation of (3), we get for its left side

$$-.966400(-\tfrac{1}{2} \times .695750 + \tfrac{1}{2} \times .602245) - .257044(\tfrac{1}{2} \times .718285 - \tfrac{1}{2} \times .798312) = -.003668.$$

Thus it is seen that the values (4) approximately satisfy conditions (1).

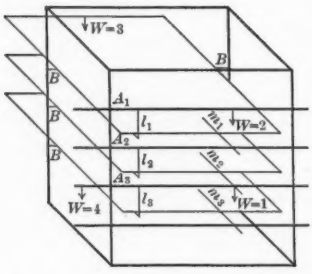
It is to be noted that in Professor Uhler's solution use is made of the law of refraction. Since this law is a special case of the Bliss-Mason condition, it follows that if Professor Uhler's results are used, the foregoing serves as a check on his solution.

II. Relating to an Equation Balance.

By E. L. Rees, University of Kentucky.

The equation balance described below, though less simple in design and construction than some others, offers, in the opinion of the writer, many compensating advantages.

This machine consists of a frame supporting two sets of levers. The levers of one set, which we shall call the scale levers, have their fulcrums at the points marked A in the figure. Just back of, and on a level with, each scale lever, except the top one, is a platform lever supported at its center on the knife edges marked B. These two sets of levers are connected in the following manner: A point one unit from the fulcrum on the first scale lever is joined to the first platform lever by a connecting rod l_1; similarly the second scale lever is connected to the second platform lever, etc. On each platform lever is a movable arm (m_1, m_2, m_3) resting on the corresponding scale lever. This arm passes through a slot in the scale lever so that all vertical forces may be transmitted through the arm to the scale lever.



The following example will serve to illustrate both the principle and the operation of the machine.

Let us solve the cubic equation $2x^3 - 3x^2 + x - 4 = 0$. Having balanced all the levers, we place on the first scale lever a weight 2, representing the first coefficient, in the equation. We then place the movable arms (m_1, m_2, m_3) in the same vertical line with this weight. Calling

its lever arm x, we see from the moments that a force of $2x$ is transmitted through the connecting rod l_1, and the arm m_1, to the point immediately below on the second scale lever. We now place a weight 3, representing the second coefficient, on the rear end of the first platform lever so that it will exert a negative or upward force on the scale lever. We now have a force $2x - 3$ applied to the second scale lever at a distance x from the fulcrum. This in turn is transferred to the third scale lever as a force of $(2x - 3)x$ to which we add a force of one unit (third coefficient) by attaching a weight 1 to the arm m_2 at this point. We also place a weight 4 (constant in equation) on the third scale lever one unit to the left of the fulcrum since the constant is negative. The resulting moment will then be $[(2x - 3)x + 1]x - 4 = 2x^3 - 3x^2 + x - 4$. We now move the weights and arms back and forth on the scale levers, keeping them in the same vertical line, until the levers are balanced. The moment will then be zero and the length of the lever arm, which is seen to be the root of the equation, may be read from the scale on the first lever. This of course gives us only positive roots. Negative roots may be found by changing the signs of the roots of the equation.

Obviously the number of scale levers corresponds to the degree of the equation of highest degree that can be solved. The figure represents a balance which may be used for solving quadratics, cubics, and quartics. In this machine the unit was taken as two inches and the working interval from zero to ten inches, thus giving roots up to five units. As the mechanical details have not yet been perfected only the general plan of construction has been given.

The satisfactory results obtained from this crudely constructed balance with which the writer experimented give assurance that a carefully constructed machine, with proper lengths of levers and unit determined experimentally, will yield results of considerable precision.

III. Relating to the Real Locus Defined by the Equation $x^y = y^x$.

By Philip Franklin, College of the City of New York.

In the September number of the Monthly (Vol. XXIII, pp. 233–237) in a discussion of the real locus defined by the equation $x^y = y^x$, E. J. Moulton raised the question of the continuity of the locus in the third quadrant.

Having met this function in another connection, I was led to consider it in the following manner:

Putting $y = mx$ in $x^y = y^x$, $x^m = mx$, from which

$$x = m^{1/(m-1)} \qquad \text{and} \qquad y = m^{m/(m-1)}.$$

The curve is easily plotted from these equations or from the equation derived from them, $r = \sec\theta \tan\theta^{1/\tan(\theta-1)}$.

In the first quadrant the curve is continuous since we obtain points on the curve for all positive values of m.

In the second and fourth quadrants the curve is discontinuous, since, in general, the $(m - 1)$th root of a negative number does not have a real value.

In the third quadrant points are obtained only for those values of m for which $m^{1/(m-1)}$ has a real negative value, which is not true in all cases; *e. g.*, when m is an even integer. The curve is, therefore, discontinuous in the third quadrant.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Dr. H. A. SAYRE, professor of mathematics at the University of Alabama, died on Dec. 2, 1916.

Dr. HENRY GUNDER, formerly professor of mathematics at Findlay College, died on Nov. 25 at the age of seventy-nine years.

The National Education Association, under the presidency of Dr. R. J. ALEY, president of the University of Maine, will meet this year at Portland, Oregon, July 7–14. The Department of Superintendence met in Kansas City on February 26 to March 3, 1917.

"On the rational, integral invariants of nilpotent algebras" is the title of a paper by Dr. OLIVE C. HAZLETT, in the December, 1916, number of the *Annals of Mathematics*.

The November number of *Genetics* contains an application of mathematics to some problems of animal husbandry under the title "Some breeding properties of the generalized Mendelian population," by E. N. WENTWORTH and B. L. REMICK, of the Kansas State Agricultural College.

The annual meeting of the Pittsburgh Section of Mathematics Teachers of the Middle States and Maryland Association was held at the University of Pittsburgh, Saturday, January 27, under the presidency of Professor C. S. ATCHISON, of Washington and Jefferson College. Papers were presented on the following subjects: "Prominent deceased mathematicians," by J. A. SILVER; "Recreations in geometry," by J. W. MORRISON; and abstracts from KEYSER'S "The Human Worth of Rigorous Thinking," by W. F. LONG.

Volume 2, No. 11, *Proceedings of the National Academy of Science*, contains mathematical contributions by Professor E. V. HUNTINGTON on "A set of independent postulates of cyclic order," and by Professor HENRY BLUMBERG on "Certain general properties of functions."

The call has been issued for the second annual meeting of the Ohio Section for the first week in April. The Executive Committee consists of Professor T. M. Focke, Case School of Applied Science, chairman, Professor C. C. Morris, Ohio State University, and Professor G. N. Armstrong, Ohio Wesleyan University, secretary.

"Calculation of the first thirty-two Eulerian numbers from central differences of zero" is the title of a paper in *The Quarterly Journal of Pure and Applied Mathematics*, Vol. 47, 1916, pp. 103–126, by S. A. JOFFE, New York City. The

Eulerian number E_n is the coefficient of $x^{2n}/(2n)!$ in the development of sec x in a power series of x. The first twenty-seven Eulerian numbers were computed by Dr. GLAZIER, *Quarterly Journal*, Vol. 45. Mr. JOFFE has verified the calculations of Dr. GLAZIER for the first twenty-seven Eulerian numbers, and has added five more numbers of the series.

The Annual Register of the American Mathematical Society for the year 1916 shows a membership of 732; during the year the attendance at general and sectional meetings numbered 490; and 205 papers were presented. The treasurer's report shows a balance of $10,198.38. The library contains 5,377 volumes.

School and Society has collected data showing that the universities of the United States granted 607 doctorates during the academic year 1915–1916. Of this number 332 degrees were granted in the sciences, 34 being in mathematics. The number receiving the doctorate in mathematics is not large, being only slightly more than 5 per cent. of the total, and is evidently not nearly sufficient to supply the demand for high-class instructors in the colleges and universities of America. If distributed geographically over the United States, but two doctors in mathematics could be supplied to every three states of the Union. The following table taken from the *Bulletin* of the American Mathematical Society shows the fourteen institutions granting the doctorates in mathematics with the names of those receiving the degree. BRYN MAWR: Mary G. Haseman; CALIFORNIA: T. A. Pierce, A. R. Williams; CHICAGO: A. F. Carpenter, A. M. Harding, W. L. Hart, J. O. Hassler, A. Henderson, A. L. Nelson, S. W. Reaves, A. R. Schweitzer, Pauline Sperry, Mary E. Wells, C. H. Yeaton; CLARK: P. Leyzerah; COLUMBIA: P. H. Linehan, F. J. McMackin; CORNELL: L. C. Cox, J. V. DePorte; HARVARD: P. M. Batchelder, R. W. Brink, A. L. Miller, N. Miller; JOHNS HOPKINS: F. D. Murnaghan, J. R. Musselman, C. H. Rawlins; MICHIGAN: W. Van N. Garretson; PENNSYLVANIA: J. R. Kline, J. H. Weaver; PRINCETON: J. W. Alexander, R. E. Gilman; VIRGINIA: E. S. Smith; WISCONSIN: T. M. Simpson; YALE: G. H. Light.

The following appointments and promotions have been announced: Dr. D. F. BARROW, of the University of Georgia, has been appointed instructor in mathematics in the Sheffield Scientific School; Dr. A. L. MILLER and Mr. R. B. BARNARD have been appointed instructors in mathematics at the University of Michigan; Dr. E. T. BELL has been promoted to an assistant professorship of mathematics at the University of Washington; Mr. C. W. WESTER has been appointed assistant professor of mathematics at the Iowa State Teachers College; Dr. P. H. LINEHAN has been promoted to an assistant professorship of mathematics at the College of the City of New York; Assistant Professor J. F. REILLY has been promoted to an associate professorship of mathematics at the University of Iowa; Dr. R. E. ROOT has been promoted to a professorship of mechanics and engineering mathematics at the U. S. Naval Academy; Assistant Professor T. FORT, of the University of Michigan, has accepted the professorship of mathematics and head of the department at the University of Alabama.

SUMMER SESSIONS.

In the March and April issues of the MONTHLY it is proposed to give a synopsis of graduate and undergraduate courses in mathematics offered by various universities and colleges during the coming summer. The importance of providing serious summer work, continuing from six to twelve weeks, is being recognized by a very large number of institutions. In so far as known to the writer, the first summer session of any university in this country was organized in 1890; since that date a large number of the institutions have taken on this broader field of usefulness, and many are providing summer curricula as complete as those offered during the traditional college year. Below are enumerated the mathematics courses so far as announcements reached the MONTHLY in time for this issue:

CORNELL UNIVERSITY. Summer session, July 9–Aug. 17. By Professor V. SNYDER: Foundations of elementary mathematics, five hours; Solutions by ruler and compass, five hours.—By Professor C. F. CRAIG: Advanced calculus, six hours.—By Professor F. W. OWENS: Projective geometry, six hours. Courses in algebra, trigonometry, analytic geometry and elementary calculus will be given.

THE UNIVERSITY OF COLORADO. Summer session, June 25–Aug. 4. By Professor A. COHEN (Johns Hopkins University): Differential Equations; Introductory course in analysis.—By Professor B. F. FINKEL (Drury College): The teaching of mathematics; Fundamental concepts of mathematics; Least squares; Fourier's series.—By Dr. G. H. LIGHT: Algebra; Trigonometry; Differential equations; Differential geometry.—Instructors have not yet been provided for courses in theory of equations, definite integrals, and theory of functions of a complex variable.

INDIANA UNIVERSITY. Summer session, June 14–August 10. By Professor S. C. DAVISSON: Theory of surfaces, five hours; Advanced calculus, five hours; Teaching of mathematics, two hours.—By Professor D. A. ROTHROCK: Calculus of variations, five hours; Solid analytic geometry, three hours.—By Professor U. S. HANNA: Differential equations, ten hours; Theory of functions, five hours. Courses in algebra, trigonometry, analytic geometry and elementary calculus are also offered.

UNIVERSITY OF MICHIGAN. Summer session, July 2–Aug. 24. By Professor W. W. BEMAN: Differential equations; Teachers' course in algebra and geometry.—By Professor J. L. MARKLEY: Functions of a complex variable; Advanced algebra.—By Professor W. B. FORD: Advanced calculus; Theory of potential.—By Professor L. C. KARPINSKI: History of mathematics.—By Professor J. W. BRADSHAW: Advanced analytics; Projective geometry.—By Dr. H. C. CARVER: Mathematical theory of finance, insurance and statistics. Courses are offered in elementary and college algebra, trigonometry, analytic geometry, and elementary calculus. Courses in modern physical and practical astronomy are offered by Professors HUSSEY, CURTISS and Dr. KIESS, of the department of astronomy.

UNIVERSITY OF KANSAS: Summer session, first term, June 7–July 18. By Professor U. G. MITCHELL: Higher algebra, three hours; Teachers' course, three hours.—By Professor E. B. STOUFFER: Differential equations, three hours; Differential calculus, three hours: Analytics of the line and circle, two hours.—By Professor H. E. JORDAN: Mathematical analysis, five hours. Second term, July 19–Aug. 15. By Professor J. N. VAN DER VRIES: Mathematical theory of investment, two hours. Courses are also announced in solid geometry, algebra and trigonometry.

THE UNIVERSITY OF CHICAGO. Summer terms, June 18–July 25 and July 26–Aug. 31. The courses offered by Professor MOORE continue during the first term, all others continue until Aug. 31. All advanced courses are given four hours per week. By Professor E. H. MOORE: The spectrum of limited infinite hermitian matrix; Series.—By Professor H. E. SLAUGHT: Differential equations.—By Professor E. J. WILCZYNSKI: Projective differential geometry.—By Professor J. W. A. YOUNG: Solid analytics.—By Professor A. C. LUNN: Relativity; Functions of a complex variable.—By Professor D. N. LEHMER (University of California): Synthetic projective geometry.—By Professor G. D. BIRKHOFF (Harvard University): Ordinary differential equations of the second order. Elementary courses are also offered in college algebra, trigonometry, analytic geometry and differential and integral calculus, and courses in the teaching of mathematics.

NORTHWESTERN UNIVERSITY. Summer session, June 25–Aug. 4. By Professor E. J. MOULTON: Teachers' course; College algebra.—By Dr. C. E. WILDER: Trigonometry; Analytic geometry; Differential calculus.

KANSAS STATE AGRICULTURAL COLLEGE. Summer session, June 23–Aug. 3. By Professor W. H. ANDREWS: Teachers' course; Integral calculus; Trigonometry.—By Professor A. E. WHITE: Differential calculus; Analytic geometry; Solid geometry. Courses are also offered in elementary algebra and plane geometry.

SYRACUSE UNIVERSITY. Summer session, July 8–Aug. 16. By Professor W. H. METZLER: Teachers' course; Solid analytic geometry.—By Professor W. G. BULLARD: Advanced algebra; Modern algebra; Solid geometry.—By Professor F. F. DECKER: Teachers' course in algebra and geometry. Courses are announced in elementary algebra and geometry, trigonometry, analytic geometry, calculus, descriptive geometry, and mechanics.

UNIVERSITY OF TEXAS. First term, June 13–July 25. By Professor M. B. PORTER: Subject matter and teaching of high school mathematics, five hours.—By Professor C. D. RICE and Dr. GOLDIE HORTON: Calculus, 15 hours.—By Professor E. L. DODD: Advanced calculus, 10 hours.—By Professor E. P. R. DUVAL: Differential equations, 5 hours. Courses in solid geometry, college algebra, trigonometry and analytic geometry are also announced. Second term, July 26–Sept. 4. By Dean H. Y. BENEDICT and Mr. H. J. ETTLINGER: 5 hours advanced work and elementary courses.

Notes on the Association.

Arrangements are in progress for the second summer meeting of the Association to be held in Cleveland, Ohio, September 6 and 7. The summer meeting of the Society will immediately precede that of the Association. A joint Committee of Arrangements has been appointed consisting of Professors F. N. Cole, W. D. Cairns, E. V. Huntington, T. M. Focke, A. D. Pitcher, and D. T. Wilson. The Program Committee for the Association consists of Professors C. S. Slichter, L. S. Hulburt, and E. J. Wilczynski.

The only part of the program thus far determined is the retiring address of past President Hedrick, in accordance with the action of the Council in New York, that each retiring president shall give an address at the next following summer meeting of the Association. Suggestions as to the program will be welcomed by the committee, and these may be sent to the chairman, Professor C. S. Slichter, University of Wisconsin, Madison, Wis.

In response to numerous questions, it will doubtless be of general interest to give some data with reference to the distribution of the votes for the officers of the Association at the last annual meeting. There were 405 votes cast, but of these 28 were unsigned, so that the location of 377 voters is definitely known. It is thought that probably most of the unsigned ballots were cast by those who voted in person at New York, but this gives no clue to the location of these voters, except that doubtless a number of them should be credited to New York, thus raising the percentage of this state as listed below.

Of the states having a membership of 40 or more, the percentages of those voting are as follows: Illinois 45%, Massachusetts 44%, Ohio 41%, California 36%, New York 34%, Missouri 33%, Pennsylvania 31%.

Of the states having a membership between 20 and 40, the percentage of those voting are as follows: Colorado 80%, Kansas 50%, Michigan 40%, Indiana 90%, New Jersey 28%, Texas 25%, Maryland 23%, Iowa 22%, Wisconsin 21%.

Of the states having a membership between 10 and 20, the percentages of those voting are as follows: Maine 43%, Virginia 41%, North Carolina 36%, Rhode Island 36%, New Hampshire 33%, Minnesota 31%, Nebraska 31%, Alabama 30%, District of Columbia 29%, Washington 21%, Georgia 21%, Canada 20%, Connecticut 16%, South Dakota 10%.

Of the states having a membership less than 10 the percentages of those voting are as follows: North Dakota 100% (one member), Utah 75%, New Mexico 66%, Arizona 50%, South Carolina 50%, Idaho 50%, Oklahoma 37%, Wyoming 33%, Arkansas, Delaware, Florida, Vermont and West Virginia each 25%, Kentucky 16%, Oregon 14%, Louisiana, Mississippi, Montana, Nevada, and Tennessee each 0%.

These figures show that the voters (and hence also the non-voters) were quite evenly distributed throughout the country. Reference to the charter

membership list will show from these percentages the actual number of voters in each state. The percentage figures, when not exact, are given to the nearest integer.

The distribution of ballots for the presidential candidates is of special interest since the result was so close. These figures show, in general, a very even distribution as between the two candidates. For instance, the presidential votes were equally (or very nearly equally) divided in the following states: Illinois, Massachusetts, Minnesota, Ohio, Pennsylvania, and New Jersey. In some states there was a clear preponderance in favor of Huntington, as in New York and Michigan, while in others there was a like preponderance for Cajori, as in California, Colorado and Missouri, but it would be difficult to discover any general geographical cleavage.

It seems probable that the full significance of the franchise was not generally understood and that the next election will bring out a much larger vote. It is hoped that the above analysis of the figures may contribute toward this end.

Applications for membership in the Association are steadily coming in. The following fourteen, received since the New York meeting, were acted on by the Council by mail vote on February 10, 1917. Ten other applications are now in hand (March 10, 1917).

R. J. ALEY, President, University of Maine, Orono, Me.

J. N. BROADLICK, Teacher, High School, Pittsburg, Kan.

J. W. CAMPBELL, Lecturer in Mathematics, Wesley College, Winnipeg, Can.

W. E. ETZEL, Professor of College Mathematics, College of St. Thomas, St. Paul, Minn.

P. E. KRETZMANN, Professor of Mathematics and Science, Concordia (Junior) College, St. Paul, Minn.

CLARENCE MCCORMICK, Instructor in Mathematics, University of Minnesota, Minneapolis, Minn.

SISTER MARY MAGNA, Head of the Science Department, St. Benedict's College, St. Joseph, Minn.

R. H. MARSHALL, Assistant Professor of Mathematics, State Normal Manual Training School, Pittsburg Kan.

FLORA PORTER, Teacher, Nashville, Tenn.

L. W. REID, Professor of Mathematics, Haverford College, Haverford, Pa.

EVAN THOMAS, Professor of Mechanics and Mathematics, University of Vermont, Burlington, Vt.

WILLIS WHITED, Engineer of Bridges, Pennsylvania State Highway Department, Harrisburg, Pa.

T. C. WOLLAN, Professor of Mathematics, Park Region Luther College, Fergus Falls, Minn.

HAVERFORD COLLEGE, Haverford, Pa., to institutional membership.

COMMUNICATIONS.

I. *To the Editors of the* MONTHLY:

Professor J. W. Young—as chairman of the National Committee on Mathematical Requirements—has requested Professor D. E. Smith and myself to coöperate in the preparation of a report on the criticisms of mathematics, making a critical examination of the grounds of the more prominent and more responsible attacks on mathematics, with a view to determining the criticisms which are clearly not valid, those which are clearly justifiable, and those concerning the validity of which there is reasonable doubt, with a view in the latter case to resolving the doubt if possible.

We desire, with your permission, to bring this matter to the attention of readers of the AMERICAN MATHEMATICAL MONTHLY, with the hope that they may assist us by bringing to our attention all possible material of value, particularly such as might otherwise escape attention. Communications on the subject may be addressed either to D. E. Smith, Teachers College, Columbia University, or to H. W. Tyler, Massachusetts Institute of Technology, Cambridge.

H. W. TYLER.

II. *To the Editors of the* MONTHLY:

In the February issue of the MONTHLY you remark (page 73) concerning "A problem in probability" by C. S. Jackson that "The proof sheets of this article never reached us from the author, having probably been lost in ocean transit." That Mr. Jackson died very suddenly on October 18, 1916, does not seem to have been earlier remarked in this country. An appeal (which has been recently circulated) on behalf of the widow and nine children left by the deceased in very straitened circumstances contains the following sentence: "So passed away one who was universally respected and esteemed, and who was held in great regard by all who knew him, both for his refined and gentle character and high intellectual attainments." The appeal is signed by officers in the Royal Military Academy at Woolwich where Mr. Jackson was mathematical master for more than a quarter of a century.

Charles Samuel Jackson took his B.A. at Cambridge, in 1889 and his M.A. in 1896. In 1894 he became a barrister at Lincoln's Inn. He was the author of a dozen short articles in *Revista de la Sociedad Española*, *Messenger of Mathematics* and *Mathematical Gazette*, and of the pamphlet entitled "The Calculus as a school subject" in a volume prepared for the International Commission on the Teaching of Mathematics. He was also joint author of five books: 1, with R. M. Milne, "A first statics" (1903); 2, with W. M. Roberts, "A first dynamics" (1909); 3, with H. C. Dunlop, "Slide rule notes" (1913); 4, with W. M. Roberts, "A book of elementary mechanics" (1914); 5, with F. J. W. Whipple and L. Roberts, "A twentieth century arithmetic" (1915).

Professor Jackson was a charter member of the Association.

R. C. ARCHIBALD.

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DISCUSSION OF FLUXIONS: FROM BERKELEY TO WOODHOUSE.[1]

By FLORIAN CAJORI, Colorado College.

The first direct statement of Newton's method and notation of fluxions was printed in 1693 in Wallis's *Algebra*. Here and in the *Principia* of 1687 Newton made use of infinitely small quantities, but in his *Quadrature of Curves* of 1704 he declared that "in the method of fluxions there is no necessity of introducing figures infinitely small." No other publication of Newton, printed either before 1704 or after, equalled the *Quadrature of Curves* in mathematical rigor. Here Newton reached his high water mark of rigidity in the exposition of fluxions. By a fluxion, Newton always meant a finite velocity. With one exception, all British writers on the new calculus before the appearance of Berkeley's *Analyst* in 1734 used the Newtonian notation consisting of dots or "prick'd letters," and also Newton's word "fluxion." But strange to say, most of these writers did not use Newton's concepts. They applied the term "fluxion" to the infinitely small quantities of Leibniz,—thus using a home label on goods of foreign manufacture. Of sixteen or more writers in Great Britain during the period of 1693–1734, nine or more call a fluxion an infinitely small quantity; three writers do not define their terms, while only four follow Newton's exposition of 1687 or 1693, involving fluxions as finite velocities and "moments" as infinitely small quantities, or else follow Newton's exposition of 1704, involving fluxions as finite velocities and avoiding infinitely small quantities almost entirely. The nine or more who used fluxions in the sense of infinitely small quantities had no hesitation in dropping quantities from an equation when they were very small in comparison with the other quantities. Altogether these writings contained a medley of philosophical doctrine which presented a great opportunity

[1] A paper read before the Mathematical Association of America at the second annual meeting, New York, December 30, 1916.